# On Strong Convergence to Equilibrium for the Boltzmann Equation with Soft Potentials

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## Contents

**1 Introduction** **2**

1.1 The Boltzmann Equation for soft and very soft potentials . . . . . . . . . . . . . . . . 3

1.2 Weak solutions of the Boltzmann equation . . . . . . . . . . . . . . . . . . . . . . . . . 6

1.3 The main results . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9

**2 Basic Lemmas Concerning Collision Integrals** **12**

**3 Construction of Weak Solutions, and the Equivalence of Two Definitions of Weak Solutions** **17**

**4 Moment Estimates for Weak Solutions** **20**

**5 Convergence to Equilibrium in Broad Generality** **28**

**6 Lower Bounds on the Convergence Rate** **33**

**7 Upper Bounds on the Convergence Rate for $\gamma \geq -1$ and Grad Angular Cutoff** **39**

**Abstract**

The paper concerns $L^1$- convergence to equilibrium for weak solutions of the spatially homogeneous Boltzmann Equation for soft potentials ($-4 \leq \gamma < 0$), with and without angular cutoff. We prove the time-averaged $L^1$-convergence to equilibrium for all weak solutions whose initial data have finite entropy and finite moments up to order greater than $2 + |\gamma|$. For the usual $L^1$-convergence we prove that the convergence rate can be controlled from below by the

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initial energy tails, and hence, for initial data with long energy tails, the convergence can be arbitrarily slow. We also show that under the integrable angular cutoff on the collision kernel with $-1 \leq \gamma < 0$, there are algebraic upper and lower bounds on the rate of $L^1$-convergence to equilibrium. Our methods of proof are based on entropy inequalities and moment estimates.

**Key words**: Boltzmann equation, soft potentials, weak solutions, strong convergence, equilibrium.

# 1 Introduction

While convergence to equilibrium for solutions of the spatially homogeneous Boltzmann equation has been extensively studied for hard potentials and Maxwellian molecules, much less is known in the case of soft potentials. For instance, for the hard sphere model, it has been proven that solutions of the equation for all initial data that have finite mass and energy always converge strongly to equilibrium at an exponential rate. The same result holds for all hard potentials with angular cutoff under only mild additional assumptions on the initial data $f_0$; e.g., that $f_0$ is square integrable. See [18] and references therein.

For Maxwellian molecules with angular cutoff, one again has exponential convergence to equilibrium if the initial data have finite moments up to order $s > 2$, although for $s = 2$ the convergence rate can be arbitrarily slow [6]. (In the Maxwellian case, as in the hard sphere case, there is no need to make any assumption such as square integrability of $f_0$, or even that the initial entropy be finite.)

For soft potentials, existing bounds [11, 20] on the rate of convergence to equilibrium are only algebraic, and so far have been obtained under certain cutoffs, not only in the angle, but also on the singularity in small relative velocities (which is present for soft potentials). It is commonly believed that for soft potentials, the convergence rate *actually is* generally worse than the former cases, and that the algebraic bounds found in the references cited above are at least qualitatively sharp.

In this paper, we show that this is indeed the case. Moreover, we also prove convergence results for a very broad class of weak solutions, and for a range of *very soft* potentials. The convergence results that we obtain at this broadest level of generality are exactly that: They assert convergence, in either a time averaged sense, or in the usual sense, but with no rate at all.

However, in the case of angular cutoff, and with a potential that is not *too soft*, we are able to prove much more: In particular, we shall show that for a natural class of weak solutions, if the initial data has moments of all orders, then the solution converges strongly in $L^1(\mathbf{R}^N, (1+|v|^2)dv)$ to its Maxwellian equilibrium at a *super-algebraic rate*; i.e., faster than any inverse power of the time $t$. In proving this result we rely in part on an entropy production inequality of Villlani [24], but also introduce a new strategy to avoid the use of pointwise lower bounds on the solution $f$ that were used in [24],

One crucial difference between hard potentials, Maxwellian molecules, and soft potentials shows up in the different behavior concerning energy tails: In the case of hard potentials with an angular cutoff, even if the initial data has no moments of order higher than 2, the solution at any strictly positive time will have moments of *all* orders [27]. That is, long energy tails, which are an obstacle to rapid convergence, are immediately eliminated for hard potentials. This is not the case for

Maxwellian molecules, but at least whatever control one has on the energy tails of the initial data is propagated *uniformly in time*. For soft potentials, the situation is much less favorable, and there is no propagation of higher moments uniformly in time. Instead, one has bounds on the growth of such moments, or uniform bounds on their time averages, as found in [10]. Such moment bounds play a crucial role in this paper, and we shall prove several new and strengthened results of this type.

Before proceeding with the introduction of our results, let us first precisely specify the equation to be studied and the notation that we shall use.

### 1.1 The Boltzmann Equation for soft and very soft potentials

The spatially homogeneous Boltzmann equation is given by (see [7, 8, 9])

$$\frac{\partial}{\partial t} f(v,t) = Q(f)(v,t)\,, \quad (v,t) \in (0,\infty) \times \mathbf{R}^N \tag{B}$$

where $N \geq 2$,

$$Q(f)(v,t) = \int_{\mathbf{R}^N \times \mathbf{S}^{N-1}} B(v - v_*, \sigma)(f' f'_* - f f_*) d\sigma dv_*\,, \tag{1.1}$$

$f = f(v,t), f' = f(v',t), f_* = f(v_*,t), f'_* = f(v'_*,t)$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|\sigma}{2}\,, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|\sigma}{2}\,, \quad \sigma \in \mathbf{S}^{N-1}$$

and $\mathbf{S}^{N-1}$ is the unit sphere in $\mathbf{R}^N$. The collision kernel $B(z,\sigma)$ is a nonnegative Borel function of $(|z|, \cos\theta)$, i.e.

$$B(z,\sigma) = B(|z|, \cos\theta)\,, \quad \cos\theta = \langle z/|z|, \sigma\rangle\,, \quad z = v - v_* \neq 0\,.$$

In the case of main physical interest, $N = 3$. Then, if the potential energy function that governs the interaction between pairs of molecules in the dilute gas is an inverse power of the distance separating them, $B$ takes the form

$$B(z,\sigma) = b(\cos\theta)|z|^\gamma \tag{1.2}$$

with the exponent $\gamma$ depending on the power in the interaction. The following ranges of $\gamma$ are distinguished [7, 8] by the methods required to treat them: The range $0 < \gamma < 1$ corresponds to *hard potentials*, $\gamma = 0$ to *Maxwellian molecules*, and $\gamma < 0$ to *soft potentials*, with the case $\gamma < -2$ corresponding to *very soft potentials* [24].

These distinctions pertain to the different strategies that must be employed in studying solutions of Eq.(B), or even interpreting it, for $\gamma$ in the different ranges. When $\gamma$ is negative, both the singularity in $B(z,\sigma)$ at $z = 0$, and the vanishing of $B(z,\sigma)$ at $z = \infty$ cause difficulties that partially account for the fact that soft potentials have been less intensively investigated than Maxwellian molecules or hard potentials. The problems caused by the vanishing of $B(z,\sigma)$ at $z = \infty$ include the fact that with soft potentials, one does not have uniform in time bounds on higher order moments of solutions; we shall return to this shortly.

The problems caused by the singularity in $B(z,\sigma)$ at $z = 0$ are more immediate: This singularity precludes a naive approach to making sense of the integral in (1.1), and hence complicates the interpretation of the equation itself.

$Q(f)$ is a difference of two integrals, and if each of them is to be integrable, it would have to be the case that

$$B(v - v_*, \sigma) f' f'_* \qquad \text{and} \qquad B(v - v_*, \sigma) f f_*$$

would both be integrable on $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$. When $B$ takes the form $B(z, \sigma) = b(\cos\theta)|z|^\gamma$ as in (1.2), with the exponent $\gamma$ and $0 < \gamma \leq 2$, at least the integration over $\mathbf{R}^3 \times \mathbf{R}^3$ poses no problem: As is well known, solutions of the Boltzmann equation (B) should conserve energy, and so an *a priori* bound on $\int_{\mathbf{R}^3} (1 + |v|^2) f(v) dv$ is natural to assume. Granted this, the integrability over $\mathbf{R}^3 \times \mathbf{R}^3$ is obvious.

There still remains the fact that for inverse power law potentials, the function $b(\cos\theta)$ in (1.2) is not integrable on $\mathbf{S}^2$, and so in many studies of Eq.(B) for hard potentials, one invokes a "Grad angular cut-off" to truncate $b(\cos\theta)$ so that it becomes integrable.

For soft potentials, the situations is more delicate: Lack of integrability of $b(\cos\theta)$ is not the only problem. When $\gamma$ is negative, the function

$$|v - v_*|^\gamma f(v) f(v_*) \tag{1.3}$$

is not in general integrable on $\mathbf{R}^3 \times \mathbf{R}^3$ under any natural hypothesis on $f$. The finite energy condition does not help, nor does the $H$-Theorem, which would justify assuming that $f \log f$ is integrable. By the Hardy-Littlewood-Sobolev inequality, the function in (1.3) would be integrable if $f$ belonged to $L^{6/(6+\gamma)}(\mathbf{R}^3)$, but there is no reason to expect control on this $L^p$ norm along any general class of solutions of Eq.(B).

To proceed, let $\varphi$ be a test function, and note that by standard formal calculations (see e.g. [21]), if we define

$$\Delta\varphi(v', v'_*, v, v_*) := \varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*) \tag{1.4}$$

and define

$$Q(f\,|\Delta\varphi)(v) = \int_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \sigma)(f' f'_* - f f_*) \Delta\varphi \, d\sigma dv_* \ ,$$

then we would have

$$\int_{\mathbf{R}^3} \varphi(v) Q(f)(v) dv = -\frac{1}{4} \int_{\mathbf{R}^3} Q(f\,|\Delta\varphi)(v) dv \, .$$

It can be shown (see Lemma 2.1 below, and the references cited there) that the following pointwise bound holds:

$$|\Delta\varphi| \leq C |v - v_*|^2 \sin\theta \ , \tag{1.5}$$

where $C$ is a constant depending on the second derivatives of $\varphi$. Moreover, if one first averages $\Delta\varphi$ with respect to the angle around the axis defined by $v - v_*$, one can improve the right hand side to

$$C |v - v_*|^2 \sin^2\theta \ .$$

For $-2 \leq \gamma \leq 0$, the factor of $|v - v_*|^2$ is enough to deal with the factor $|v - v_*|^\gamma$ in (1.3), and thus – neglecting for the moment problems with $b(\cos\theta)$ – the bound (1.5) provides what is needed to make sense of a weak form of Eq.(B) for $\gamma$ in this range.

The analysis of this case was initiated by Arkeryd [2], who actually considered only $-1 \leq \gamma < 0$, and it was carried forward by a number of authors. See [21] for a discussion of the history.

The case $\gamma < -2$ is more subtle; there is nothing more to be squeezed out of $\Delta\varphi$ to help with the singularity at $z = 0$. Results in this very soft range were first obtained by Villani [21]. A key idea in his work is to use an additional regularity estimate on the solutions $f$ coming not from the entropy itself, but from the *entropy production.* Later, we shall return to this point in more detail. Hopefully now at least it is clear where the distinction between soft and very soft potentials comes from.

There is still the problem that for inverse power law interactions, the function $b(\cos\theta)$ is not integrable on $\mathbf{S}^2$. The problem comes from a singularity in the small $\theta$ collisions; i.e., the grazing collisions. Whenever one wishes to consider $Q(f)$ as a difference of two separate integrals – the *gain* and *loss* terms – it is necessary to impose a *Grad angular cut-off* which is the assumption that $b(\cos\theta)$ *is* integrable on $\mathbf{S}^2$.

However, for many purposes, this is unnecessary, and one can takes advantage of the weak form $Q(f|\Delta\varphi)$ and the extra factors of $\sin\theta$ in (1.5) and the bound below it. This takes care of the singularity in $b(\cos\theta)$ for $-3 < \gamma < 1$, since in this case one has

$$\int_{\mathbf{S}^2} B(z,\sigma)\sin^2\theta\, d\sigma = \text{const.}|z|^\gamma < \infty\ .$$

The case $\gamma = -3$ is the Coulomb potential, and is therefore of particular interest. However, in this case $B(z,\sigma) = C_0(\sin(\theta/2))^{-4}|z|^{-3}$, so that

$$\int_{\mathbf{S}^2} B(z,\sigma)\sin^2\theta\, d\sigma = \infty\ .$$

This difficulty with the Coulomb interaction is a genuine part of the physics, and not a weakness of current technical tools. Without an angular cut-off, the Boltzmann equation does not make sense for the Coulomb interaction. See [21] for further discussion of this, and what is done in plasma physics to study the kinetics of plasmas nonetheless.

Here, we stay within the framework of the Boltzmann equation (B) with $N \geq 2$, and often will simply require of $b(\cos\theta)$ the mild cut-off hypothesis that $b(\cos\theta)\sin^2\theta$ is integrable on $\mathbf{S}^{N-1}$:

In this paper we shall write that $B(z,\sigma)$ satisfies a *mild angular cut-off*, provided

$$\int_{\mathbf{S}^{N-1}} B(z,\sigma)\sin^2\theta\, d\sigma \leq A^*|z|^\gamma\,, \qquad -4 \leq \gamma < 0 \tag{1.6}$$

for some constant $0 < A^* < \infty$. Again, the difference between this and the stronger Grad angular cut-off is the factor of $\sin^2\theta$ in the integral, and the possibility of making such a mild cut-off assumption in the context of weak solutions has been exploited by a number of authors; again we refer to [21] for an account of the history.

In addition to the upper bound in (1.6), we shall also sometimes need to invoke a corresponding lower bound. For instance, to prove the moment estimates mentioned above, and prove the convergence to equilibrium, we assume in addition that $B(z,\sigma) > 0$ for almost every $(z,\sigma) \in \mathbf{R}^N \times \mathbf{S}^{N-1}$ and there is a constant $0 < A_* < \infty$ such that

$$\int_{\mathbf{S}^{N-1}} B(z,\sigma)\sin^2\theta d\sigma \geq A_*(1+|z|^2)^{\gamma/2}\,, \qquad -4 \leq \gamma < 0\,. \tag{1.7}$$

### 1.2 Weak solutions of the Boltzmann equation

Having explained the difference between soft and very soft potentials, and the kinds of cut-off assumptions we shall consider, we are ready to introduce the class of weak solutions of Eq.(B) that we shall study.

Eq.(B) for soft potentials is usually investigated by entropy and moment methods with working spaces of Lebesgue measurable functions $f : \mathbf{R}^N \to \mathbf{R}$

$$L^1_0(\mathbf{R}^N) = L^1(\mathbf{R}^N), \quad L^1_s(\mathbf{R}^N) = \left\{ f \;\middle|\; \|f\|_{L^1_s} := \int_{\mathbf{R}^N} \langle v\rangle^s |f(v)| dv < \infty \right\}, \qquad s \in \mathbf{R},$$
$$L^1_s\log L(\mathbf{R}^N) = \left\{ f \;\middle|\; \int_{\mathbf{R}^N} \langle v\rangle^s |f(v)|(1 + |\log|f(v)||) dv < \infty \right\}$$

where and throughout the paper we use the notation

$$\langle v\rangle = (1 + |v|^2)^{1/2}\,.$$

The entropy (Boltzmann $H$-functional) and the entropy dissipation are given by

$$H(f) = \int_{\mathbf{R}^N} f(v)\log f(v) dv\,, \qquad 0 \le f \in L^1\log L(\mathbf{R}^N)\,,$$

$$D(f) = \frac{1}{4}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(v - v_*, \sigma)(f' f'_* - f f_*)\log\left(\frac{f' f'_*}{f f_*}\right) d\sigma dv_* dv\,. \tag{1.8}$$

Here and below we define $(a - b)\log(a/b) = \infty$ if $a > b = 0$ or $b > a = 0$; $(a - b)\log(a/b) = 0$ if $a = b = 0$.

As noted above, in order to establish a weak form of Eq.(B), we need to be able to make sense of the expression

$$\int_{\mathbf{S}^{N-1}} B(|v - v_*|, \cos\theta)\Delta\varphi\, d\sigma$$

even when, due to the singularity in $B$ at $\theta = 0$, the integrand is not integrable. As we shall see in Section 2, this can be done under smoothness assumptions on $\varphi$ provided we *first* integrate over all the variables in $\mathbf{S}^{N-1}$ except $\theta$, and *then* integrate over $\theta$. That, is with $\mathbf{k} = (v - v_*)/|v - v_*|$, we can parameterize $\mathbf{S}^{N-1}$ by $(\theta, \omega) \in [0, \pi] \times \mathbf{S}^{N-2}(\mathbf{k})$ through $\sigma = \cos(\theta)\mathbf{k} + \sin(\theta)\omega$. Using this parameterization, and interpreting the integral as an iterated integral, we shall show in Section 2 that when $\varphi$ is sufficiently smooth, integrating first in $\omega$ renders the $\theta$ integral convergent. On this basis (see Section 2 for details), we define for all $\varphi \in C^2(\mathbf{R}^N)$

$$L[\Delta\varphi](v, v_*) = \int_0^\pi B(|v - v_*|, \cos\theta)\sin^{N-2}\theta \left(\int_{\mathbf{S}^{N-2}(\mathbf{k})} \Delta\varphi\, d\omega\right) d\theta \tag{1.9}$$

where $\Delta\varphi = \Delta\varphi(v', v'_*, v, v_*)$ is given by (1.4).

The relevant space $\mathcal{T}$ of test functions $\varphi$ for which this construction works is given by

$$\mathcal{T} = \left\{ \varphi \in C^2(\mathbf{R}^N) \;\middle|\; \sup_{v\in\mathbf{R}^N} \left(\langle v\rangle^{-2}|\varphi(v)| + \langle v\rangle^{-1}|\partial\varphi(v)| + |\partial^2\varphi(v)|\right) < \infty \right\},$$

where $\partial\varphi(v) = (\partial_{v_1}\varphi(v), ..., \partial_{v_N}\varphi(v))$, $\partial^2\varphi(v) = \left(\partial^2_{v_iv_j}\varphi(v)\right)_{N\times N}$, $|\partial\varphi(v)| = \left(\sum_{1\le i\le N} |\partial_{v_i}\varphi(v)|^2\right)^{1/2}$ and $|\partial^2\varphi(v)| = \left(\sum_{1\le i,j\le N} |\partial^2{}_{v_iv_j}\varphi(v)|^2\right)^{1/2}$.

As in the previous subsection, we also define, here for all $\varphi \in \mathcal{T}$,

$$Q(f\,|\Delta\varphi)(v) = \int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} B(v-v_*,\sigma)(f'f'_* - ff_*)\Delta\varphi\, d\sigma dv_*\ .$$

By formal calculation we have

$$\int_{\mathbf{R}^N} \varphi(v)Q(f)(v)dv = -\frac{1}{4}\int_{\mathbf{R}^N} Q(f\,|\Delta\varphi)(v)dv = \frac{1}{2}\int_{\mathbf{R}^N\times\mathbf{R}^N} L[\Delta\varphi](v,v_*)ff_*dv_*dv\,.$$

Referring to Arkeryd [2] and Goudon [13] (for $-1 \le \gamma < 0$ and $-2 \le \gamma < 0$ respectively) and Villani [21] ( for $-4 < \gamma < 0$), we introduce

**Definition of Weak Solutions**. *Suppose the kernel $B$ satisfies (1.6). Let $0 \le f_0 \in L^1_2 \cap L^1\log L^1(\mathbf{R}^N)$. A nonnegative measurable function $f(v,t)$ on $\mathbf{R}^N\times[0,\infty)$ is called a weak solution of Eq.(B) with $f(v,0) = f_0(v)$ if the following* (i), (ii) *hold:*

(i) $f \in L^\infty([0,\infty); L^1_2 \cap L^1\log L^1(\mathbf{R}^N))$ *and*

$$H(f(t)) + \int_0^t D(f(\tau))d\tau \le H(f_0)\,, \quad t \ge 0\,. \tag{1.10}$$

(ii) *For all $\varphi \in \mathcal{T}$, if $-4 \le \gamma < -2$, then*

$$\int_{\mathbf{R}^N} \varphi(v)f(v,t)dv = \int_{\mathbf{R}^N} \varphi(v)f_0(v)dv - \frac{1}{4}\int_0^t d\tau\int_{\mathbf{R}^N} Q(f\,|\Delta\varphi)(v,\tau)dv\,, \quad t \ge 0\,; \tag{1.11}$$

*and if $-2 \le \gamma < 0$, then*

$$\int_{\mathbf{R}^N} \varphi(v)f(v,t)dv = \int_{\mathbf{R}^N} \varphi(v)f_0(v)dv + \frac{1}{2}\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N} L[\Delta\varphi]ff_*dv_*dv\,, \quad t \ge 0\,. \tag{1.12}$$

Note that the particular functions $\varphi(v) = 1, v_i\,(i = 1,2,...,N)$ and $|v|^2$ all belong to $\mathcal{T}$ and satisfy $\Delta\varphi \equiv 0$. So the above definition implies that *every weak solution $f$ of Eq.(B) conserves the mass, momentum and energy, i.e.*

$$\int_{\mathbf{R}^N}(1,v,|v|^2)f(v,t)dv = \int_{\mathbf{R}^N}(1,v,|v|^2)f_0(v)dv\,, \quad t \ge 0\,.$$

It will be seen that the collision integrals in (ii) are absolutely convergent with respect to the total measure $d\sigma dv_*dvd\tau$ and $dv_*dvd\tau$ respectively (see Lemma 2.2 below). For very soft potentials, $-4 \le \gamma < -2$, this is essentially due to the entropy inequality (1.10) as first noted in [21]; the corresponding weak solutions are also called $H$-solutions.

We shall prove in Section 3 that the integral equations (1.11) and (1.12) are both equivalent to a full and common version like (1.11) with $\varphi \in C_b^1(\mathbf{R}^N \times [0,\infty)) \cap L^\infty([0,\infty); C_b^2(\mathbf{R}^N))$ where

$$C_b^2(\mathbf{R}^N) = \left\{ \varphi \in C^2(\mathbf{R}^N) \,\middle|\, \sup_{v \in \mathbf{R}^N} (|\varphi(v)| + |\partial\varphi(v)| + |\partial^2\varphi(v)|) < \infty \right\}.$$

The precise statement of this equivalence is given in the following proposition:

**Proposition 1.1**. *Suppose the kernel $B$ satisfies (1.6). Let $0 \le f_0 \in L_2^1 \cap L^1\log L^1(\mathbf{R}^N)$, $0 \le f \in L^\infty([0,\infty); L_2^1 \cap L\log L(\mathbf{R}^N))$ satisfy the entropy inequality (1.10) and $f|_{t=0} = f_0$. Then the following are equivalent (for total range $-4 \le \gamma < 0$):*

*(a) $f$ is a weak solution of Eq.(B).*

*(b) $f$ satisfies the equation (1.11) for all $\varphi \in C_b^2(\mathbf{R}^N)$.*

*(c) $f$ satisfies the following equation: For all $\varphi \in C_b^1(\mathbf{R}^N \times [0,\infty)) \cap L^\infty([0,\infty); C_b^2(\mathbf{R}^N))$*

$$\int_{\mathbf{R}^N} \varphi(v,t) f(v,t) dv = \int_{\mathbf{R}^N} \varphi(v,0) f_0(v) dv + \int_0^t d\tau \int_{\mathbf{R}^N} \frac{\partial \varphi(v,\tau)}{\partial \tau} f(v,\tau) dv$$

$$- \frac{1}{4} \int_0^t d\tau \int_{\mathbf{R}^N} Q(f\,|\Delta\varphi)(v,\tau) dv\,, \qquad t \ge 0\,. \tag{1.13}$$

The existence of weak solutions has been proven respectively by Arkeryd [2] for $-1 \le \gamma < 0$, Goudon [13] for $-2 \le \gamma < 0$, and Villani [21] for $-4 < \gamma < 0$. In Proposition 1.2 below we summarize these results, with one improvement: We also treat the case $\gamma = -4$.

**Proposition 1.2**. *Let $B(z,\sigma)$ satisfy (1.6). Then for any $0 \le f_0 \in L_2^1 \cap L^1\log L(\mathbf{R}^N)$, the Eq.(B) has a weak solution $f$ satisfying $f|_{t=0} = f_0$.*

We shall provide a proof of Proposition 1.2 in Section 3 below. Despite the fact that apart from the case $\gamma = -4$, a proof can be found in the references cited above, there are motivations for presenting the details here.

First, the only paper covering the range $-4 < \gamma < -2$ is Villani's [21], and he bases his analysis on the relation between the Boltzmann equation and the Landau equation. In fact, he gives a complete proof for the case of the Landau equation for $-4 < \gamma < 0$, and then simply discusses the main ideas of proof for the Boltzmann equation. While the discussion is quite clear, and while there are good physical reasons for making a connection with the Landau equation, it is possible to proceed somewhat more directly for the Boltzmann equation, as we do here: Our proof is direct, relatively short, complete and covers the case $\gamma = -4$. (In [21], the hypothesis $-4 < \gamma$, was used two times for Landau equation, and hence needed for Boltzmann equation.)

A second reason for presenting a proof here is that to go beyond $\gamma = -2$, one must use entropy production estimates. We shall use simple entropy production arguments systematically throughout the paper, not only to construct weak solutions. But using them to construct weak solutions for very soft potentials provides an excellent topic with which to introduce them.

Finally, various approximation procedures that are used in the proof of existence are also used in our study of convergence to equilibrium, and for this reason it is quite useful to have them included explicitly in this paper.

### 1.3 The main results

By changing scales one can assume without loss of generality that initial data have unite mass, zero momentum and unit temperature, i.e.

$$f_0 \in L^1_{(1,0,1)}(\mathbf{R}^N) := \left\{ 0 \le f \in L^1_2(\mathbf{R}^N) \,\middle|\, \int_{\mathbf{R}^N} (1, v, \frac{1}{N}|v|^2) f(v) dv = (1,0,1) \right\}.$$

The Maxwellian in $L^1_{(1,0,1)}(\mathbf{R}^N)$ is given by

$$M(v) = (2\pi)^{-N/2} \exp(-|v|^2/2), \quad v \in \mathbf{R}^N. \tag{1.14}$$

To study $L^1$- convergence to equilibrium, we shall use a property that the $L^1$-distances $\|f - M\|_{L^1}$ and $\|f - M\|_{L^1_2}$ are almost equivalent [6]: There is an explicit constant $0 < C_N < \infty$ depending only on $N$, such that

$$\|f - M\|_{L^1_2} \le C_N \|f - M\|_{L^1} \log\left( \frac{6}{\|f - M\|_{L^1}} \right) \qquad \forall\, f \in L^1_{(1,0,1)}(\mathbf{R}^N).$$

This implies that ( with a different $C_N < \infty$)

$$\|f - M\|_{L^1} \le \|f - M\|_{L^1_2} \le C_N \sqrt{\|f - M\|_{L^1}} \qquad \forall\, f \in L^1_{(1,0,1)}(\mathbf{R}^N). \tag{1.15}$$

Our main results are Theorems 1-3 below; their proofs will be given in latter sections.

**Theorem 1**. *Let $B(z,\sigma)$ satisfy (1.6) and (1.7). For any initial datum $f_0 \in L^1_{(1,0,1)} \cap L^1_s \cap L^1\mathrm{log}L(\mathbf{R}^N)$ with $s > 2$, let $f(v,t)$ be a weak solution of Eq.(B) with $f|_{t=0} = f_0$ . Then*

(I) (*Moment Estimates*).

$$\|f(t)\|_{L^1_s} \le C_s(1+t), \qquad \frac{1}{t}\int_0^t \|f(\tau)\|_{L^1_{s+\gamma}} d\tau \le C_s, \qquad \forall\, t > 0 \tag{1.16}$$

*where the constant $0 < C_s < \infty$ depends only on $N, \gamma, A_*, A^*, s$ and $\|f_0\|_{L^1_s}$, and in case $-4 \le \gamma < -2$, $C_s$ depends also on $H(f_0)$.*

(II) (*Time Averaged Convergence*). *If $s > 2 + |\gamma|$, then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|f(t) - M\|_{L^1_2} dt = 0 \tag{1.17}$$

*where $M \in L^1_{(1,0,1)}(\mathbf{R}^N)$ is the Maxwellian (1.14).*

**Remarks:** (1) The moment estimates in (1.16) were first established by Desvillettes [10] under the Grad angular cut-off on $B$ with $-1 < \gamma < 0$. Villani [21] then proved (1.16) under the mild cut-off assumption (1.6)-(1.7) for $-4 < \gamma < 0$ and $s \le 4$, and in [23] he concluded further that $\forall\, s > 2$, $\exists\, \lambda_s > 0$ such that $\|f(t)\|_{L^1_s} \le C_s(1+t)^{\lambda_s}$. Here we prove that $\lambda_s \equiv 1$ for all $s > 2$ (thanks to the integrability $\int_0^\infty D(f(t))dt < \infty$).

(2) *Theorem 2 provides the first convergence results for weak solutions for very soft potentials without any cut-off, or with very weak angular cutoff.* (Recall that (1.6) holds for free if $\gamma > -3$). For the usual $L^1$ convergence, it has been proven in [11] and [20] that $\|f(t) - M\|_{L^1} \le C(1+t)^{-\lambda}$ ($\lambda > 0$) only under quite strong cut-off assumptions soft potentials; e.g., that $z \mapsto B(z,\sigma)$ is bounded near $z = 0$.

Our next results concern lower and upper bounds of convergence rate to equilibrium for certain classes of solutions of Eq.(B). We show that a general lower bound can be obtained for such initial data $f_0\in L^1_{(1,0,1)}\cap L^1_s\cap L^1\log L(\mathbf{R}^N)$ that have energy long-tails:

$$\limsup_{R\to\infty} R^{\beta}\int_{|v|>R}|v|^2 f_0(v)dv=\infty \tag{1.18}$$

where $\beta=\min\{s,\, s-2+|\gamma|\}$ for $s\ge 2$, $-4\le\gamma<0$. Note that the condition (1.18) implies that for any constant $K>0$, the equation

$$(\mathrm{R}(t))^{\beta}\int_{|v|>\mathrm{R}(t)}|v|^2 f_0(v)dv=K(1+t)^{2-[2/s]}\,,\quad t\in[0,\infty) \tag{1.19}$$

has a minimal solution $\mathrm{R}(t)>0$. Here $[x]$ denotes the largest integer not exceeding $x$.

**Theorem 2**. *Let $f_0\in L^1_{(1,0,1)}\cap L^1_s\cap L^1\log L(\mathbf{R}^N)$ satisfy (1.18) for some $s\ge 2$, and let $f(v,t)$ be a weak solution of Eq.(B) with initial datum $f|_{t=0}=f_0$. Then*

(I) *For any constant $K_0\in(0,\infty)$ there exists $K\in[K_0,\infty)$ which depends only on $N,\gamma,A^*,A_*,s,\|f_0\|_{L^1_s},H(f_0)$ and $K_0$ such that for the minimal solution $\mathrm{R}(t)$ of (1.19) we have*

$$\|f(t)-M\|_{L^1_2}\ge\int_{|v|>\mathrm{R}(t)}|v|^2 f_0(v)dv\qquad\forall\, t\ge 0\,. \tag{1.20}$$

*As a consequence we have the following explicit lower bounds:*

(II) *Suppose $s>2$, $\beta=\min\{s\,,\,s-2+|\gamma|\}$, and there are constants $s-2<\delta<\beta$ and $0<\varepsilon_0\,,R_0<\infty$ such that*

$$f_0(v)\ge\varepsilon_0\langle v\rangle^{-(N+2+\delta)} \tag{1.21}$$

*for all $|v|\ge R_0$. Then there is a computable constant $C>0$ such that*

$$\|f(t)-M\|_{L^1_2}\ge C\,(1+t)^{-\lambda}\qquad\forall\, t\ge 0 \tag{1.22}$$

*where $\lambda=2\delta/(\beta-\delta)$.*

(III) *Suppose $s=2$, $\beta=\min\{2\,,\,|\gamma|\}$. Let $A\in C^1_b([0,\infty))$ satisfy*

$$\lim_{t\to\infty}A(t)=0\,,\quad \inf_{t\ge 0}(1+t)^{\delta}A(t)>0\,,\quad A_1(t):=-\frac{d}{dt}A(t)\ge 0\quad\text{on}\quad[0,\infty) \tag{1.23}$$

*where $0<\delta<\beta$. Suppose for some $0<\varepsilon_0,R_0<\infty$*

$$f_0(v)\ge\varepsilon_0|v|^{-(N+1)}A_1(|v|)\qquad\forall\,|v|\ge R_0\,. \tag{1.24}$$

*Then there are constants $0<c,C<\infty$ such that*

$$\|f(t)-M\|_{L^1_2}\ge CA(c\,t^{\alpha})\qquad\forall\, t\ge 0 \tag{1.25}$$

*where $\alpha=1/(\beta-\delta)$.*

There are many initial data $f_0$ that satisfy all conditions in Theorem 2. For example, in the case $s=2$, one can choose

$$A(t)=(1+t)^{-\delta}\,,\quad [1+\log(1+t)]^{-1}\,,\quad [1+\log(1+\log(1+t))]^{-1}\,,...$$

which means that the rate of convergence to equilibrium can be arbitrarily slow for $s=2$. This fact has been observed in [6] for Maxwellian molecules ($\gamma=0$) with angular cutoff. Note also that for any initial datum $f_0\in L^1_{(1,0,1)}(\mathbf{R}^N)$, the mass tail of $f_0$ always decays at least with algebraic order 2, $\int_{|v|>R}f_0(v)dv\le NR^{-2}$, but the energy tail $\int_{|v|>R}|v|^2f_0(v)dv$ may decay very slowly. This is why we consider the energy tail (hence $L^1_2$ norm) rather than the mass tail.

We now turn to upper bounds on the rate of convergence. Here, we must impose more restrictive conditions on the collision kernel: We assume that $B(z,\sigma)$ satisfies the following cutoff conditions (with constant $K_*>0$):

$$K_*(1+|z|^2)^{\gamma/2}\le B(z,\sigma)\le b(\cos\theta)|z|^{\gamma}\,,\quad -1\le\gamma<0 \tag{1.26}$$

$$A_0:=|\mathbf{S}^{N-2}|\int_0^{\pi}b(\cos\theta)\sin^{N-2}\theta\,d\theta<\infty\,. \tag{1.27}$$

**Theorem 3**. *Let $B(z,\sigma),\gamma$ satisfy (1.26)-(1.27) and let $f_0\in L^1_{(1,0,1)}\cap L^1_s\mathrm{log}L^1(\mathbf{R}^N)$ with $s>10$. Then there exist a finite constant $C$ and a weak solution $f(v,t)$ of Eq.(B) with $f|_{t=0}=f_0$ such that*

$$\|f(t)-M\|_{L^1_2}\le C\,(1+t)^{-\lambda}\,,\qquad t\ge 0 \tag{1.28}$$

*where*

$$\lambda=\frac{s-10}{12}>0\,. \tag{1.29}$$

**Remark:** In this theorem we do not assume that $f_0$ has any strictly positive pointwise lower bounds, nor shall we make use of any pointwise lower bounds on the weak solutions.

In Theorem 3, if the initial datum satisfies (1.21), then the corresponding solution $f$ satisfies both (1.22) and (1.28), i.e., the convergence rate to equilibrium satisfies both upper and lower bounds that are only algebraic:

$$C_1(1+t)^{-\lambda_1}\le\|f(t)-M\|_{L^1_2}\le C\,(1+t)^{-\lambda}\,,\qquad t\ge 0\,.$$

One of the main tools we use to prove Theorem 3 is an entropy production bound of Villani [24] for *super hard* potentials; i.e., $\gamma=+2$. As Villani showed in [24], for super hard potentials, there is an especially nice inequality relating the entropy production and the relative entropy. And moreover, while super hard potentials are themselves non-physical, one can use the super hard entropy production bound to obtain entropy production bounds for physically interesting hard potentials, using *moment bounds* and *pointwise lower bounds* on the solutions.

Our Theorem 1 provides moment bounds for soft potentials that are good enough to proceed with an adaptation of this part of Villani's argument to soft potentials, but the pointwise lower bounds are more problematic in this setting.

The pointwise lower bounds enter Villani's argument as follows: To estimate the entropy production $D(f)$ for $\gamma<2$ in terms of $\mathcal{D}_2(f)$, the entropy production for $\gamma=2$, a simple Hölder argument explained in Section 7 leads to the consideration of the quatitiy

$$\mathcal{D}_k(f)=\frac{1}{4}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}}(1+|v-v_*|^2)^{k/2}(f'f'_*-ff_*)\log\left(\frac{f'f'_*}{ff_*}\right)d\sigma dvdv_*$$

for $k > 2$. It is easy to see that $\mathcal{D}_k(f)$ can be estimated in terms of $L^1$ bounds on $\langle v\rangle^k f(v)\log f(v)$, and clearly the negative part of this function is integrable if $f$ satisfies a bound of the type $f(v) \geq Ce^{-c|v|^p}$ and $f$ has moments of sufficiently high orders. The details are somewhat more complicated than indicated in this sketch, but the sketch should nonetheless give a fair indication of the interplay between moment bounds and pointwise lower bounds in Villani's arguments. While suitable pointwise lower bounds are available for the hard potentials that Villani considers, they are not available for soft potentials, at least not for the sort of general initial data that we wish to consider.

A novel element in our proof of Theorem 3 is a strategy for avoiding any pointwise bounds which is explained in Section 7. Given a solution $f(v,t)$ of Eq.(B), we define the function $g(v,t)$ by

$$g(v,t) = (1-e^{-t-1})f(v,t) + e^{-t-1}M(v) \ .$$

Evidently, $g(v,t)$ has good pointwise lower bounds by construction. Although $g$ is not itself a solution to Eq.(B), it is closely related enough to one, namely $f$, that we shall use Villani's entropy production inequality for super hard potentials to prove and an entropy production inequality relating $D(g)$ and $H(g|M)$ and show that $H(g|M)$ tends to zero at any polynomial rate provided the initial datum $f_0$ has sufficiently many moments.

The most technically involved part of the proof is the demonstration that the $L^1$ norm of $\langle v\rangle^k g(v,t)\log g(v,t)$ is bounded by a constant multiple of $(1+t)^2$, again , provided the initial data has sufficiently many moments. This approach to proving and using such *entropic moment estimates* is one of the more novel features of this paper, and may well have other applications. The condition $\gamma \geq -1$ is used in this part of the proof of Theorem 3.

We thank the referee of this paper who encouraged us to work harder on the proof of Theorem 3, and relax the stronger conditions on the initial data that we had imposed in a previous version, and we thank this referee for his many other useful remarks and suggestions as well.

## 2 Basic Lemmas Concerning Collision Integrals

In this section we collect some lemmas that will be used to ensure integrability of certain collision integrals, as well as to estimate others in terms of entropy dissipation.

There are nothing fundamentally new in this section except some quantitative improvements and mild generalizations of some lemmas that can be found in previous works by Goudon [13] and Villani [21], and references they cite.

The first lemma justifies the definition of $L[\Delta\varphi](v,v_*)$ that figures in our definition of weak solutions. We begin with a more complete explanation of the notation used in the definition of $L[\Delta\varphi](v,v_*)$ that we have given in the introduction.

Let

$$\mathbf{k} = \frac{v-v_*}{|v-v_*|} \quad \text{if} \quad v \neq v_* \,; \quad \mathbf{k} = \mathbf{e}_1 = (1,0,...,0) \quad \text{if} \quad v = v_* \,.$$

Under the spherical coordinate transform $\sigma = \cos\theta\,\mathbf{k} + \sin\theta\,\omega\,, \theta \in [0,\pi]\,, \omega \in \mathbf{S}^{N-2}(\mathbf{k})$ we have

$$\begin{cases} v' = \cos^2(\theta/2)v + \sin^2(\theta/2)v_* + \frac{1}{2}|v-v_*|\sin\theta\,\omega\,, \\ \\ v_*' = \sin^2(\theta/2)v + \cos^2(\theta/2)v_* - \frac{1}{2}|v-v_*|\sin\theta\,\omega \end{cases} \qquad \omega \in \mathbf{S}^{N-2}(\mathbf{k}) \tag{2.1}$$

$$|v'-v| = |v'_*-v_*| = |v-v_*|\sin(\theta/2)\,,\quad |v'-v_*| = |v'_*-v| = |v-v_*|\cos(\theta/2)\,. \tag{2.2}$$

Here

$$\mathbf{S}^{N-2}(\mathbf{k}) = \{\omega\in\mathbf{S}^{N-1}\mid \langle\omega,\mathbf{k}\rangle = 0\,\}\quad (\,N\ge 3)\,;\qquad \mathbf{S}^0(\mathbf{k}) = \{-\mathbf{k}^\perp\,,\,\mathbf{k}^\perp\}\quad (N=2)$$

where $\mathbf{k}^\perp\in\mathbf{S}^1$ satisfies $\langle\mathbf{k}^\perp,\mathbf{k}\rangle = 0$. Also, for any $F\in L^1(\mathbf{S}^{N-1})$ or any measurable function $F\ge 0$ on $\mathbf{S}^{N-1}$, we have

$$\int_{\mathbf{S}^{N-1}} F(\sigma)d\sigma = \int_0^\pi \sin^{N-2}\theta\left(\int_{\mathbf{S}^{N-2}(\mathbf{k})} F(\cos\theta\mathbf{k}+\sin\theta\,\omega)d\omega\right)d\theta$$

and in case $N=2$ we define

$$\int_{\mathbf{S}^0(\mathbf{k})} g(\omega)d\omega = g(-\mathbf{k}^\perp)+g(\mathbf{k}^\perp)\,.$$

Let $|\mathbf{S}^{N-2}(\mathbf{k})| = \int_{\mathbf{S}^{N-2}(\mathbf{k})}d\omega$, etc. Then $|\mathbf{S}^{N-2}(\mathbf{k})| = |\mathbf{S}^{N-2}|$ for $N\ge 3$, $|\mathbf{S}^0(\mathbf{k})| = |\mathbf{S}^0| = 2$ for $N=2$.

**Lemma 2.1**. *Let $\varphi\in C^2(\mathbf{R}^N)\,,\Delta\varphi = \varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*)$. Then for all $\sigma\in\mathbf{S}^{N-1}$, $v,v_*\in\mathbf{R}^N$*

$$|\Delta\varphi|\le 2^{(4-3m)/2}\Big(\sup_{|u|\le\sqrt{|v|^2+|v_*|^2}}|\partial^m\varphi(u)|\Big)|v-v_*|^m\sin\theta\,,\qquad m=1,2\,; \tag{2.3}$$

$$\frac{1}{|\mathbf{S}^{N-2}|}\left|\int_{\mathbf{S}^{N-2}(\mathbf{k})}\Delta\varphi\,d\omega\right|\le\Big(\sup_{|u|\le\sqrt{|v|^2+|v_*|^2}}|\partial^2\varphi(u)|\Big)|v-v_*|^2\sin^2\theta\,. \tag{2.4}$$

**Proof**. Observing that $-\sigma = \cos(\pi-\theta)\,\mathbf{k}+\sin(\pi-\theta)(-\omega)$ and $\Delta\varphi$ is invariant under the reflection $\sigma\to-\sigma$, we can assume without loss of generality that $\theta\in[0,\pi/2]$. In this case we have $\sin(\theta/2)\le(\sin\theta)/\sqrt{2}$.

By writing $\Delta\varphi = (\varphi'-\varphi)+(\varphi'_*-\varphi_*)$ one sees that (2.3) for $m=1$ follows from the first equality in (2.2). Next writing $\Delta\varphi = (\varphi'-\varphi)-(\varphi_*-\varphi'_*)$ and using $v_*-v'_* = v'-v$, we compute

$$\begin{aligned}\Delta\varphi &= \int_0^1\langle\partial\varphi(v+t(v'-v))-\partial\varphi(v'_*+t(v'-v)),v'-v\rangle dt\\ &= \int_0^1\!\!\int_0^1(v-v'_*)\partial^2\varphi(\xi_{t,\tau})(v'-v)^T d\tau dt\end{aligned}$$

with $|\xi_{t,\tau}|\le\max\{|v|,|v'|,|v_*|,|v'_*|\}\le\sqrt{|v|^2+|v_*|^2}$. Since $|v'_*-v||v'-v| = \frac{1}{2}|v-v_*|^2\sin\theta$, this gives (2.3) for $m=2$. To prove (2.4) we write $\Delta\varphi = (\varphi'-\varphi)+(\varphi'_*-\varphi_*)$ and use $v'_*-v_* = -(v'-v)$. Then

$$\begin{aligned}&\Delta\varphi = \langle\partial\varphi(v)-\partial\varphi(v_*),v'-v\rangle\\ &+\int_0^1(1-t)(v'-v)\partial^2\varphi(v+t(v'-v))(v'-v)^T dt\\ &+\int_0^1(1-t)(v'_*-v_*)\partial^2\varphi(v_*+t(v'_*-v_*))(v'_*-v_*)^T dt\,.\end{aligned}$$

Since by (2.1)

$$\frac{1}{|\mathbf{S}^{N-2}|}\int_{\mathbf{S}^{N-2}(\mathbf{k})}\langle\partial\varphi(v)-\partial\varphi(v_*),v'-v\rangle d\omega=\langle\partial\varphi(v)-\partial\varphi(v_*),v_*-v\rangle\sin^2(\theta/2)$$

where we used $\int_{\mathbf{S}^{N-2}(\mathbf{k})}\langle\partial\varphi(v)-\partial\varphi(v_*),\omega\rangle d\omega=0$, it follows that

$$\frac{1}{|\mathbf{S}^{N-2}|}\left|\int_{\mathbf{S}^{N-2}(\mathbf{k})}\Delta\varphi\,d\omega\right|\le 2\Big(\sup_{|u|\le\sqrt{|v|^2+|v_*|^2}}|\partial^2\varphi(u)|\Big)\,|v-v_*|^2\sin^2(\theta/2)\,.$$

□

Our next lemma provides bounds on certain collision integrals in terms of entropy dissipation. As in the case of the local Sobolev bounds on the collision kernel first proved by Lions [15], and then extended in subsequent work [22, 1], the proof of our bounds depends on the pointwise inequality (2.10) below. However, as our bounds do not involve local Sobolev norms, the proof is somewhat simpler.

**Lemma 2.2**. *Let $B=B(v-v_*,\sigma)$ satisfy (1.6), $0\le f\in L^1_2(\mathbf{R}^N)$ satisfy $D(f)<\infty$. Then:*

(I) *For any nonnegative measurable function $\Psi$ on $\mathbf{R}^N\times\mathbf{R}^N$ satisfying*

$$\Psi(v',v_*')=\Psi(v,v_*)\quad\forall\,(v,v_*,\sigma)\in\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}$$

*we have*

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}}B\,\Psi(v,v_*)\sin\theta\,|f'f_*'-ff_*|d\sigma dv_*dv$$
$$\le\left(4A^*\int_{\mathbf{R}^N\times\mathbf{R}^N}[\Psi(v,v_*)]^2|v-v_*|^\gamma ff_*dv_*dv\right)^{1/2}\sqrt{D(f)}\tag{2.5}$$

*where $A^*$ is the constant in (1.6).*
*(*II*) Let $m\in\{1,2\}$ be such that $0\le 2m+\gamma\le 2$. Then*

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}}B|v-v_*|^m\sin\theta|f'f_*'-ff_*|d\sigma dv_*dv\le\sqrt{4A^*}\|f\|_{L^1_2}\sqrt{D(f)}\tag{2.6}$$

*and consequently for all $\varphi\in C^2_b(\mathbf{R}^N)$*

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}}B|\Delta\varphi||f'f_*'-ff_*|d\sigma dv_*dv\le\sqrt{8A^*}\|\partial^m\varphi\|_{L^\infty}\|f\|_{L^1_2}\sqrt{D(f)}\,.\tag{2.7}$$

*(*III*) For all $\varphi\in\mathcal{T}$, if $-4\le\gamma<-2$, then*

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}}B|\Delta\varphi||f'f_*'-ff_*|d\sigma dv_*dv\le\sqrt{A^*}\|\partial^2\varphi\|_{L^\infty}\|f\|_{L^1_2}\sqrt{D(f)}\tag{2.8}$$

*and if $-2\le\gamma<0$, then*

$$\int_{\mathbf{R}^N\times\mathbf{R}^N}\int_0^\pi B(|v-v_*|,\cos\theta)\sin^{N-2}\theta\left|\int_{\mathbf{S}^{N-2}(\mathbf{k})}\Delta\varphi\,d\omega\right|d\theta\,ff_*dv_*dv\le A^*\|\partial^2\varphi\|_{L^\infty}\|f\|^2_{L^1_2}\tag{2.9}$$

*Proof.* Applying the elementary inequality

$$|a-b| \le (\sqrt{a}+\sqrt{b})\sqrt{\frac{1}{4}(a-b)\log(\frac{a}{b})}\,, \quad a,b\ge 0 \tag{2.10}$$

to $a=f'f'_*, b=ff_*$ and using Cauchy-Schwarz inequality we have

$$\begin{aligned}&\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\,\Psi(v,v_*)\sin\theta\,|f'f'_*-ff_*|d\sigma dv_* dv\\ &\le \left(4\int_{\mathbf{R}^N\times\mathbf{R}^N}\left(\int_{\mathbf{S}^{N-1}} B\,\sin^2\theta\,d\sigma\right)[\Psi(v,v_*)]^2 ff_*\,dv_*dv\right)^{1/2}\sqrt{D(f)}\end{aligned}$$

which gives (2.5) by assumption (1.6). The condition $0\le 2m+\gamma\le 2$ implies $|v-v_*|^{2m+\gamma}\le \langle v\rangle^2\langle v_*\rangle^2$. So applying (2.5) to $\Psi(v,v_*)=|v-v_*|^m$ gives (2.6). The inequality (2.7) follows from (2.3),(2.4) and (2.6). The inequality (2.8) follows from (2.3) and (2.6) with $m=2$. Finally from (2.4) and $|\mathbf{S}^{N-2}|\int_0^\pi B(|v-v_*|,\cos\theta)\sin^N\theta d\theta\le A^*$ we have

$$\int_0^\pi B(|v-v_*|,\cos\theta)\sin^{N-2}\theta\left|\int_{\mathbf{S}^{N-2}(\mathbf{k})}\Delta\varphi\,d\omega\right|d\theta \le \|\partial^2\varphi\|_{L^\infty}A^*|v-v_*|^{2+\gamma}$$

and $|v-v_*|^{2+\gamma}\le\langle v\rangle^2\langle v_*\rangle^2$ when $-2\le\gamma<0$. This gives (2.9). □

The last lemma in this section justifies the equalities resulting from formal calculation that are cited just above the definition of weak solutions, at least for certain cutoff parts of the collision integrals – which is just what we shall need in the next section.

**Lemma 2.3**. *Suppose $B(z,\sigma)$ satisfies (1.6). For any $\lambda>0$, let*

$$B^\lambda(z,\sigma)=\mathbf{1}_{\{|z|\le\lambda\}}B(z,\sigma)\,,\quad B_\lambda(z,\sigma)=\mathbf{1}_{\{|z|>\lambda\}}B(z,\sigma) \tag{2.11}$$

*and let $Q(\cdot), Q^\lambda(\cdot), L_\lambda[\cdot]$ be the operators corresponding to the kernels $B(z,\sigma), B^\lambda(z,\sigma)$ and $B_\lambda(z,\sigma)$ respectively. Then for all $0\le f\in L^1_2(\mathbf{R}^N)$ satisfying $D(f)<\infty$ we have*

(I) *If $-4\le\gamma<-2$ then for any $\varphi\in\mathcal{T}$ and any $\lambda>0$*

$$\int_{\mathbf{R}^N} Q(f\,|\Delta\varphi)(v)dv=\int_{\mathbf{R}^N} Q^\lambda(f\,|\Delta\varphi)(v)dv-2\int_{\mathbf{R}^N\times\mathbf{R}^N} L_\lambda[\Delta\varphi]ff_*dv_*dv\,. \tag{2.12}$$

(II) *If $-2\le\gamma<0$ then for all $\varphi\in C^2_b(\mathbf{R}^N)$*

$$\int_{\mathbf{R}^N} Q(f\,|\Delta\varphi)(v)dv=-2\int_{\mathbf{R}^N\times\mathbf{R}^N} L[\Delta\varphi]ff_*dv_*dv\,. \tag{2.13}$$

*Proof.* (I) Suppose $-4\le\gamma<-2$. Given $\varphi\in\mathcal{T}$. By Lemma 2.1 we have $|\Delta\varphi(v',v'_*,v,v_*)|\le \|\partial^2\varphi\|_{L^\infty}|v-v_*|^2\sin\theta$. So applying (2.6) with $m=2$ gives

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(v-v_*,\sigma)|\Delta\varphi||f'f'_*-ff_*|d\sigma dv_*dv<\infty \tag{2.14}$$

and thus

$$\int_{\mathbf{R}^N} Q(f\,|\Delta\varphi)(v)dv=\int_{\mathbf{R}^N} Q^\lambda(f\,|\Delta\varphi)(v)dv+\int_{\mathbf{R}^N} Q_\lambda(f\,|\Delta\varphi)(v)dv \tag{2.15}$$

with $Q^{\lambda}(\cdot), Q_{\lambda}(\cdot)$ corresponding to $B^{\lambda}z,\sigma)$ and $B_{\lambda}(z,\sigma)$. Introduce further truncation

$$B_{\lambda,\varepsilon}(v-v_*,\sigma)=\mathbf{1}_{\{\sin\theta>\varepsilon\}}B_{\lambda}(v-v_*,\sigma)\,,\qquad \varepsilon>0$$

and let $Q_{\lambda,\varepsilon}(\cdot), L_{\lambda,\varepsilon}[\cdot]$ correspond to $B_{\lambda,\varepsilon}(z,\sigma)$. Then using (2.14) and dominated convergence we have

$$\int_{\mathbf{R}^N} Q_{\lambda}(f\,|\Delta\varphi)(v)dv=\lim_{\varepsilon\to 0+}\int_{\mathbf{R}^N} Q_{\lambda,\varepsilon}(f\,|\Delta\varphi)(v)dv\,. \tag{2.16}$$

By Lemma 2.1 and $B_{\lambda,\varepsilon}\le \frac{1}{\varepsilon}B_{\lambda}\sin\theta$ we have

$$B_{\lambda,\varepsilon}(v-v_*,\sigma)|\Delta\varphi|\le \frac{\|\partial^2\varphi\|_{L^\infty}}{2\varepsilon}|v-v_*|^2\mathbf{1}_{\{|v-v_*|>\lambda\}}B(v-v_*,\sigma)\sin^2\theta\,.$$

Since $|v-v_*|^{2+\gamma}\mathbf{1}_{\{|v-v_*|>\lambda\}}\le\lambda^{2+\gamma}$, it follows that

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B_{\lambda,\varepsilon}|\Delta\varphi|ff_*d\sigma dv_*dv\le\frac{A^*}{2\varepsilon}\|\partial^2\varphi\|_{L^\infty}\lambda^{2+\gamma}\|f\|^2_{L^1}<\infty\,.$$

This allows us to use the standard derivation and obtain

$$\int_{\mathbf{R}^N} Q_{\lambda,\varepsilon}(f\,|\Delta\varphi)(v)dv=-2\int_{\mathbf{R}^N\times\mathbf{R}^N} L_{\lambda,\varepsilon}[\Delta\varphi]ff_*dv_*dv\,. \tag{2.17}$$

Also by Lemma 2.1

$$\begin{aligned}&|L_{\lambda}[\Delta\varphi](v,v_*)|\,,\ \ \sup_{\varepsilon>0}|L_{\lambda,\varepsilon}[\Delta\varphi](v,v_*)|\\ &\le\int_0^{\pi}B_{\lambda}(|v-v_*|,\cos\theta)\sin^{N-2}\theta\left|\int_{\mathbf{S}^{N-2}(\mathbf{k})}\Delta\varphi\,d\omega\right|d\theta\le A^*\|\partial^2\varphi\|_{L^\infty}\lambda^{2+\gamma}\,.\end{aligned}$$

Therefore using dominated convergence gives

$$\lim_{\varepsilon\to 0+}\int_{\mathbf{R}^N\times\mathbf{R}^N} L_{\lambda,\varepsilon}[\Delta\varphi]ff_*dv_*dv=\int_{\mathbf{R}^N\times\mathbf{R}^N} L_{\lambda}[\Delta\varphi]ff_*dv_*dv\,.$$

This together with (2.15), (2.16) and (2.17) proves (2.12).

(II) Suppose $-2\le\gamma<0$. Consider $B_{\varepsilon}(z,\sigma)=\mathbf{1}_{\{\sin\theta>\varepsilon\}}B(z,\sigma)$ and let $Q_{\varepsilon}(\cdot)$, $L_{\varepsilon}[\cdot]$ correspond to $B_{\varepsilon}(z,\sigma)$. For any $\varphi\in C^2_b(\mathbf{R}^N)$ we have, by Lemma 2.2 (use (2.7) with $m=1$) and dominated convergence, that

$$\int_{\mathbf{R}^N} Q(f\,|\,\Delta\varphi)(v)dv=\lim_{\varepsilon\to 0+}\int_{\mathbf{R}^N} Q_{\varepsilon}(f\,|\,\Delta\varphi)(v)dv\,. \tag{2.18}$$

As shown above using $B_{\varepsilon}\le\frac{1}{\varepsilon}B\sin\theta$ and $|v-v_*|^{2+\gamma}\le\langle v\rangle^2\langle v_*\rangle^2$ we have

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B_{\varepsilon}|\Delta\varphi|ff_*d\sigma dv_*dv<\infty$$

hence

$$\int_{\mathbf{R}^N} Q_{\varepsilon}(f\,|\,\Delta\varphi)(v)dv=-2\int_{\mathbf{R}^N\times\mathbf{R}^N} L_{\varepsilon}[\Delta\varphi](v,v_*)ff_*dv_*dv\ . \tag{2.19}$$

Since $0\le B_{\varepsilon}\le B$ and $B_{\varepsilon}\to B$ $(\varepsilon\to 0)$ pointwise, it follows from (2.9) and dominated convergence that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^N\times\mathbf{R}^N} L_{\varepsilon}[\Delta\varphi]ff_*dv_*dv=\int_{\mathbf{R}^N\times\mathbf{R}^N} L[\Delta\varphi]ff_*dv_*dv\,.$$

This together with (2.18) and (2.19) proves (2.13). $\Box$

# 3 Construction of Weak Solutions, and the Equivalence of Two Definitions of Weak Solutions

In this section we prove Proposition 1.1 (equivalence) and Proposition 1.2 (existence).

**Proof of Proposition 1.1.** First, "$(a) \Longrightarrow (b)$" is a consequence of $C_b^2(\mathbf{R}^N) \subset \mathcal{T}$ and part (II) of Lemma 2.3. To prove "$(b) \Longrightarrow (a)$", we consider approximation. Let $\chi \in C_c^\infty(\mathbf{R}^N)$ satisfy

$$0 \le \chi \le 1 \quad \text{on} \quad \mathbf{R}^N, \quad \chi(v) = 1 \quad \forall\, |v| \le 1\,; \quad \chi(v) = 0 \quad \forall\, |v| > 2\,. \tag{3.1}$$

Given any $\varphi \in \mathcal{T}$. Let $\varphi_n(v) = \varphi(v)\chi(v/n)$, $n \ge 1$. It is easily seen that $\{\varphi_n\} \subset C_b^2(\mathbf{R}^N)$ and

$$\sup_{n\ge 1}\sup_{v\in\mathbf{R}^N}\big(\langle v\rangle^{-2}|\varphi_n(v)| + \langle v\rangle^{-1}|\partial\varphi_n(v)| + |\partial^2\varphi_n(v)|\big) < \infty.$$

Since $|\varphi_n(v)| \le |\varphi(v)|$ and $\varphi_n(v) \to \varphi(v)\ \ \forall\, v \in \mathbf{R}^N$, it follows that

$$\int_{\mathbf{R}^N}\varphi(v)f(v,t)dv = \int_{\mathbf{R}^N}\varphi(v)f_0(v)dv - \frac{1}{4}\lim_{n\to\infty}\int_0^t d\tau\int_{\mathbf{R}^N} Q(f\,|\,\Delta\varphi_n)(v,\tau)dv\,, \qquad t \ge 0\,.$$

Assume $-4 \le \gamma < -2$. Since

$$\Delta\varphi_n \to \Delta\varphi \quad (\,n\to\infty\,) \quad \forall\,(v,v_*,\sigma) \in \mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}$$

and $\sup\limits_{n\ge 1}|\Delta\varphi_n| \le C_\varphi|v-v_*|^2\sin\theta$, it follows from part (III) of Lemma 2.2 and dominated convergence that

$$\lim_{n\to\infty}\int_0^t d\tau\int_{\mathbf{R}^N} Q(f\,|\,\Delta\varphi_n)(v,\tau)dv = \int_0^t d\tau\int_{\mathbf{R}^N} Q(f\,|\,\Delta\varphi)(v,\tau)dv\,.$$

Next assume $-2 \le \gamma < 0$. By part (II) of Lemma 2.3 we have

$$-\int_0^t d\tau\int_{\mathbf{R}^N} Q(f\,|\,\Delta\varphi_n)(v,\tau)dv = 2\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N} L[\Delta\varphi_n]ff_*dv_*dv$$

Since $\lim\limits_{n\to\infty} L[\Delta\varphi_n](v,v_*) = L[\Delta\varphi](v,v_*)$ and

$$|L[\Delta\varphi](v,v_*)|\,,\ \sup_{n\ge1} L[\Delta\varphi_n](v,v_*)| \le C_\varphi|v-v_*|^{2+\gamma} \le C_\varphi\langle v\rangle^{2+\gamma}\langle v_*\rangle^{2+\gamma}$$

it follows from dominated convergence that

$$-\lim_{n\to\infty}\int_0^t d\tau\int_{\mathbf{R}^N} Q(f\,|\,\Delta\varphi_n)dv = 2\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N} L[\Delta\varphi]ff_*dv_*dv\,.$$

Therefore $f$ is a weak solution.

Now we are going to prove "$(b) \Longleftrightarrow (c)$". "$(b) \Longleftarrow (c)$" is trivial. To prove "$(b) \Longrightarrow (c)$" we denote for notation convenience that

$$Q(f\,|\Delta\varphi(s,\cdot))(v,\tau) = Q(f(\tau)\,|\Delta\varphi(s))(v)\ , \qquad \int_{\mathbf{R}^N} g(t)dv = \int_{\mathbf{R}^N} g(v,t)dv\,.$$

Given any $\varphi \in C_b^1(\mathbf{R}^N\times[0,\infty)) \cap L^\infty([0,\infty); C_b^2(\mathbf{R}^N))$. By Lemma 2.1 and Lemma 2.2 there is $m \in \{1,2\}$ such that

$$|\Delta\varphi(v',v_*'v,v_*,t)| \le C_\varphi|v-v_*|^m\sin\theta\,, \tag{3.2}$$

$$\int_{t_1}^{t_2} ds \int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\,|v-v_*|^m \sin\theta |f'f'_* - ff_*| d\sigma dv_* dv \le C_f \int_{t_1}^{t_2} \sqrt{D(f(s))}\, ds \tag{3.3}$$

for all $0 \le t_1 < t_2 < \infty$, where $C_\varphi = 2\sup_{t\ge 0} \|\partial_v^m \varphi(\cdot,t)\|_{L^\infty}$, $C_f = \sqrt{4A^*}\sup_{t\ge 0}\|f(t)\|_{L^1_2}$. Applying (1.11) to the test function $v\mapsto \varphi(v,t_2)$ we obtain

$$\int_{\mathbf{R}^N} \varphi(t_2) f(t_2) dv = \int_{\mathbf{R}^N} \varphi(t_2) f(t_1) dv - \frac{1}{4}\int_{t_1}^{t_2} d\tau \int_{\mathbf{R}^N} Q(f(\tau)\,|\Delta\varphi(t_2)) dv\,.$$

This gives

$$\begin{aligned}&\int_{\mathbf{R}^N} \varphi(t_2) f(t_2) dv - \int_{\mathbf{R}^N} \varphi(t_1) f(t_1) dv\\ &= \int_{\mathbf{R}^N} \left(\varphi(t_2)-\varphi(t_1)\right) f(t_1) dv - \frac{1}{4}\int_{t_1}^{t_2} d\tau \int_{\mathbf{R}^N} Q(f(\tau)\,|\Delta\varphi(t_2)) dv\end{aligned}$$

which implies by (3.2) and (3.3) that $t\mapsto \int_{\mathbf{R}^N}\varphi(t)f(t)dv$ is continuous on $[0,\infty)$. Choose $t_1 = s,\, t_2 = s+h, 0<h<1$. Taking integration with respect to $s\in[0,t]$ and changing variables we compute

$$\begin{aligned}&\frac{1}{h}\int_t^{t+h}\int_{\mathbf{R}^N} \varphi(s) f(s) dv ds - \frac{1}{h}\int_0^h \int_{\mathbf{R}^N} \varphi(s) f(s) dv ds\\ &= \int_0^t ds \int_{\mathbf{R}^N} \frac{1}{h}\left(\varphi(s+h)-\varphi(s)\right) f(s) dv - \frac{1}{4} I(t,h)\,,\end{aligned}$$

$$\begin{aligned}I(t,h) &:= \int_0^1 d\tau \int_0^t ds \int_{\mathbf{R}^N} Q(f(s+\tau h)\,|\Delta\varphi(s+h)) dv\\ &= \int_0^1 d\tau \int_0^{t+1} ds \int_{\mathbf{R}^N} \mathbf{1}_{\{\tau h\le s\le t+\tau h\}} Q(f(s)\,|\Delta\varphi(s+(1-\tau)h)) dv\,.\end{aligned}$$

Since

$$\mathbf{1}_{\{\tau h\le s\le t+\tau h\}}\Delta\varphi(v',v'_*v,v_*,s+(1-\tau)h) \to \mathbf{1}_{\{0\le s\le t\}}\Delta\varphi(v',v'_*v,v_*,s) \quad (h\to 0)$$

for almost every $(v,v_*,\sigma,s,\tau)\in \mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}\times[0,t+1]\times[0,1]$, it follows from (3.2), (3.3) and dominated convergence that

$$I(t,h) \to \int_0^t ds \int_{\mathbf{R}^N} Q(f(s)\,|\Delta\varphi(s)) dv \qquad (\,h\to 0\,)\,.$$

Therefore

$$\begin{aligned}&\int_{\mathbf{R}^N} \varphi(t) f(t) dv - \int_{\mathbf{R}^N} \varphi(0) f_0 dv\\ &= \int_0^t ds \int_{\mathbf{R}^N} (\partial_s\varphi(s)) f(s) dv - \frac{1}{4}\int_0^t ds \int_{\mathbf{R}^N} Q(f(s)\,|\Delta\varphi(s)) dv\end{aligned}$$

for all $t\in(0,\infty)$. Hence, $f$ satisfies (1.13). $\square$

**Proof of Proposition 1.2**. As usual, we shall use approximate solutions. For every $n\in\mathbf{N}$, let $B_n = \min\{B\,,\,n\}$ and let $Q_n(\cdot), D_n(\cdot)$ and $L_n[\cdot]$ correspond to the kernel $B_n$. It is well-known that

for every $n$ there is a unique strong (or mild) solution $f^n(v,t)$ of Eq.(B) with the kernel $B_n$ and the initial datum $f^n|_{t=0} = f_0$. And $f^n(v,t)$ conserves the mass, momentum, and energy and satisfies the entropy inequality

$$H(f^n(t)) + \int_0^t D_n(f^n(\tau))d\tau \le H(f_0)\,, \quad t \ge 0\,. \tag{3.4}$$

These imply $\sup\limits_{n\ge1\,,\,t\ge0} \int_{\mathbf{R}^N} f^n(v,t)(\langle v\rangle^2 + |\log f^n(v,t)|)dv < \infty$.

Since $f^n$ are also weak solutions, they satisfy equation (1.11) which together with (2.7) and (3.4) imply that for all $\varphi \in C_b^2(\mathbf{R}^N)$ and all $|t_1 - t_2| \le 1$

$$\sup_{n\ge1}\left|\int_{\mathbf{R}^N}\varphi(v)f^n(v,t_1)dv - \int_{\mathbf{R}^N}\varphi(v)f^n(v,t_2)dv\right| \le C\|\partial^2\varphi\|_{L^\infty}|t_1-t_2|^{1/2}\,. \tag{3.5}$$

Here $C$ depends only on $A^*, \|f_0\|_{L^1_2}$ and $H(f_0)$. From this we have for any $\psi \in L^\infty(\mathbf{R}^N)$

$$\sup_{|t_1-t_2|\le\delta}\sup_{n\ge1}\left|\int_{\mathbf{R}^N}\psi(v)f^n(v,t_1)dv - \int_{\mathbf{R}^N}\psi(v)f^n(v,t_2)dv\right| \to 0 \quad \text{as} \quad \delta \to 0+\ .$$

By a standard argument, there exist a subsequence of $\{f^n\}$ (still denoted by $\{f^n\}$), and a $(v,t)$-measurable function $0 \le f \in L^\infty([0,\infty); L^1_2 \cap L^1\log L(\mathbf{R}^N))$ such that

$$\forall\, t \ge 0 \qquad f^n(\cdot,t) \rightharpoonup f(\cdot,t) \quad \text{weakly in} \quad L^1(\mathbf{R}^N) \tag{3.6}$$

Hence, by convexity and Fatou's Lemma, we conclude from (3.4) that $f$ satisfies the entropy inequality (1.10). (The details of such an argument may be found in [12].) Thus, to prove that $f$ is a weak solution, it only needs to show that for any $\varphi \in C_b^2(\mathbf{R}^N)$, $t \in [0,\infty)$,

$$\lim_{n\to\infty}\int_0^t d\tau\int_{\mathbf{R}^N} Q_n(f^n\,|\,\Delta\varphi)dv = \int_0^t d\tau\int_{\mathbf{R}^N} Q(f\,|\,\Delta\varphi)dv\,. \tag{3.7}$$

To do this, we use the following property (which is a consequence of weak convergence (3.6) and $\sup_{n\ge1,\,t\ge0}\|f^n(t)\|_{L^1_2} < \infty$): If for some $\alpha < 2$, $\Phi(v,v_*)$ and $\Phi_n(v,v_*)$ satisfy

$$|\Phi(v,v_*)|\,,\ \sup_{n\ge1}|\Phi_n(v,v_*)| \le C\langle v\rangle^\alpha\langle v_*\rangle^\alpha\,,$$

$$\Phi_n(v,v_*) \to \Phi(v,v_*) \quad (\,n\to\infty\,) \quad \text{a.e. } (v,v_*) \in \mathbf{R}^N\times\mathbf{R}^N$$

then

$$\lim_{n\to\infty}\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}\Phi_n f^n f^n_* dv_*dv = \int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}\Phi f f_* dv_*dv\,. \tag{3.8}$$

Suppose $-2 \le \gamma < 0$. Then

$$|L[\Delta\varphi](v,v_*)|\,,\ \sup_{n\ge1}|L_n[\Delta\varphi](v,v_*)| \le C_\varphi\langle v\rangle^{2+\gamma}\langle v_*\rangle^{2+\gamma}\ ,$$

so applying the relation (2.13) in Lemma 2.3, we see that the convergence (3.7) follows from (3.8) with $\Phi(v,v_*) = L[\Delta\varphi](v,v_*)$ and $\Phi_n(v,v_*) = L_n[\Delta\varphi](v,v_*)$.

Next, for $-4 \le \gamma < -2$, we truncate: For any $\lambda > 0$, let $B_{n,\lambda} = \mathbf{1}_{\{|v-v_*|>\lambda\}} B_n$ , $B_\lambda = \mathbf{1}_{\{|v-v_*|>\lambda\}} B$ and let $Q_{n,\lambda}(\cdot), L_{n,\lambda}[\cdot]$ and $Q_\lambda(\cdot), L_\lambda[\cdot]$ correspond to $B_{n,\lambda}$ and $B_\lambda$ respectively. Then

$$\begin{aligned}
&\int_0^t d\tau \int_{\mathbf{R}^N} Q_n(f^n \,|\, \Delta\varphi)dv - \int_0^t d\tau \int_{\mathbf{R}^N} Q(f \,|\, \Delta\varphi)dv \\
&= \int_0^t d\tau \int_{\mathbf{R}^N} Q_n(f^n \,|\, \Delta\varphi)dv - \int_0^t d\tau \int_{\mathbf{R}^N} Q_{n,\lambda}(f^n \,|\, \Delta\varphi)dv \\
&+ \int_0^t d\tau \int_{\mathbf{R}^N} Q_{n,\lambda}(f^n \,|\, \Delta\varphi)dv - \int_0^t d\tau \int_{\mathbf{R}^N} Q_\lambda(f \,|\, \Delta\varphi)dv \\
&+ \int_0^t d\tau \int_{\mathbf{R}^N} Q_\lambda(f \,|\, \Delta\varphi)dv - \int_0^t d\tau \int_{\mathbf{R}^N} Q(f \,|\, \Delta\varphi)dv \\
&:= I_{n,\lambda}(t) + J_{n,\lambda}(t) + I_\lambda(t)\,.
\end{aligned}$$

Using part (I) of Lemma 2.2 we have

$$\begin{aligned}
|I_{n,\lambda}(t)| &\le C_\varphi \int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\mathbf{1}_{\{|v-v_*|\le\lambda\}}|v-v_*|^2 \sin\theta |f^{n\prime} f_*^{n\prime} - f^n f_*^n| d\sigma dv_* dv \\
&\le C_\varphi \left( \int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N} \mathbf{1}_{\{|v-v_*|\le\lambda\}}|v-v_*|^{4+\gamma} f^n f_*^n dv_* dv \right)^{1/2} \left( \int_0^t D_n(f^n(\tau))d\tau \right)^{1/2} \\
&\le C_\varphi \left( \int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N} \mathbf{1}_{\{|v-v_*|\le\lambda\}} f^n f_*^n dv_* dv \right)^{1/2} .
\end{aligned}$$

By $\sup\limits_{n\ge1,\tau\ge0} \int_{\mathbf{R}^N} f^n(v,\tau)|\log f^n(v,\tau)|dv < \infty$ we obtain for all $0 < \lambda < 1 < R$

$$\sup_{n\ge1,\tau\ge0} \int_{\mathbf{R}^N} f^n(v,\tau) \left( \int_{|v-v_*|\le\lambda} f^n(\tau, v_*)dv_* \right) dv \le C\left( R\,\lambda^N + \frac{1}{\log R} \right) \|f_0\|_{L^1} .$$

This implies that $\sup_{n\ge1} |I_{n,\lambda}(t)| \to 0$ as $\lambda \to 0$. Similarly $I_\lambda(t) \to 0\,(\lambda \to 0)$. To estimate $J_{n,\lambda}(t)$ we use (2.12) in Lemma 2.3 to get

$$|J_{n,\lambda}(t)| = 2\left| \int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N} L_{n,\lambda}[\Delta\varphi] f^n f_*^n dv_* dv - \int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N} L_\lambda[\Delta\varphi] f f_* dv_* dv \right| .$$

Since $L_{n,\lambda}[\Delta\varphi](v,v_*) \to L_\lambda[\Delta\varphi](v,v_*)\,(n\to\infty)$ for all $(v,v_*) \in \mathbf{R}^N \times \mathbf{R}^N$ and

$$|L_\lambda[\Delta\varphi](v,v_*)|\,,\ \sup_{n\ge1} |L_{n,\lambda}[\Delta\varphi](v,v_*)| \le C_\varphi \mathbf{1}_{\{|v-v_*|>\lambda\}} |v-v_*|^{2+\gamma} \le C_{\varphi,\lambda}$$

it follows from (3.8) that $J_{n,\lambda}(t) \to 0\,(n\to\infty)\ \forall\,\lambda > 0$. These imply (3.7) for $-4 \le \gamma < -2$. Therefore, $f$ is a weak solution. $\square$

# 4 Moment Estimates for Weak Solutions

In this section we prove the first part of Theorem 1; i.e., the moment estimates. We need two lemmas that provide estimates of $\Delta\varphi$ for $\varphi(v) = \langle v\rangle^s$. These are so-called Povzner type estimates, but including averaging that allows them to be applied in our weak solution setting.

For any $v, v_* \in \mathbf{R}^N$ let $\mathbf{h} = \frac{v+v_*}{|v+v_*|}$ for $v+v_* \neq 0$ and $\mathbf{h} = \mathbf{e}_1 = (1, 0, ..., 0)$ for $v + v_* = 0$; $\mathbf{k} = \frac{v-v_*}{|v-v_*|}$ for $v - v_* \neq 0$ and $\mathbf{k} = \mathbf{e}_1$ for $v - v_* = 0$. Then using representation (2.2) we have with $\omega \in \mathbf{S}^{N-2}(\mathbf{k})$

$$|v'|^2 = \frac{|v|^2 + |v_*|^2}{2} + \frac{|v+v_*||v-v_*|}{2}\left(\langle \mathbf{h}, \mathbf{k}\rangle \cos\theta + \sqrt{1-\langle \mathbf{h}, \mathbf{k}\rangle^2}\,\sin\theta\,\langle \mathbf{j}, \omega\rangle\right),$$

$$|v'_*|^2 = \frac{|v|^2 + |v_*|^2}{2} - \frac{|v+v_*||v-v_*|}{2}\left(\langle \mathbf{h}, \mathbf{k}\rangle \cos\theta + \sqrt{1-\langle \mathbf{h}, \mathbf{k}\rangle^2}\,\sin\theta\,\langle \mathbf{j}, \omega\rangle\right)$$

where $\mathbf{j} = \frac{\mathbf{h} - \langle \mathbf{h}, \mathbf{k}\rangle \mathbf{k}}{\sqrt{1-\langle \mathbf{h}, \mathbf{k}\rangle^2}}$ for $|\langle \mathbf{h}, \mathbf{k}\rangle| < 1$ and $\mathbf{j} = \mathbf{e}_1$ for $|\langle \mathbf{h}, \mathbf{k}\rangle| = 1$.

To prove the moment estimates we need the following lemmas:

**Lemma 4.1.** *For all $s > 2$ and $v, v_* \in \mathbf{R}^N$*

$$\begin{aligned}&\frac{1}{|\mathbf{S}^{N-2}|}\int_{\mathbf{S}^{N-2}(\mathbf{k})}\left(\langle v'\rangle^s + \langle v'_*\rangle^s - \langle v\rangle^s - \langle v_*\rangle^s\right)d\omega\\ &\le s(s-2)\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{(s-4)/2}|v+v_*|^2|v-v_*|^2\left[(1-\langle \mathbf{h}, \mathbf{k}\rangle^2)^{\bar{s}/2} - 2^{-s/2-3}\right]\sin^2\theta\end{aligned}$$

*where* $\bar{s} = \min\{s-2,\, 2\}$.

*Proof.* Let

$$\rho = \frac{1}{2}\left(\langle v\rangle^2 + \langle v_*\rangle^2\right), \quad r = \frac{|v+v_*||v-v_*|}{\langle v\rangle^2 + \langle v_*\rangle^2}, \quad X = r\langle \mathbf{h}, \mathbf{k}\rangle, \quad Y = r\sqrt{1-\langle \mathbf{h}, \mathbf{k}\rangle^2}\,.$$

Then from the representation of $|v'|^2, |v'_*|^2$ we have

$$\begin{aligned}\langle v'\rangle^2 &= \rho\,(1 + X\cos\theta\, + Y\sin\theta\,\langle \mathbf{j}, \omega\rangle\,)\,, \quad \langle v\rangle^2 = \rho(1+X)\,,\\ \langle v'_*\rangle^2 &= \rho\,(1 - X\cos\theta\, - Y\sin\theta\,\langle \mathbf{j}, \omega\rangle\,)\,, \quad \langle v_*\rangle^2 = \rho(1-X)\end{aligned}$$

and so

$$W(v, v_*, \theta) := \rho^{-k}\int_{\mathbf{S}^{N-2}(\mathbf{k})}\left(\langle v'\rangle^{2k} + \langle v'_*\rangle^{2k} - \langle v\rangle^{2k} - \langle v_*\rangle^{2k}\right)d\omega$$

$$= \int_{\mathbf{S}^{N-2}(\mathbf{k})}\left\{\sum_{i=1,-1}\left(1 + iX\cos\theta + iY\sin\theta\,\langle \mathbf{j}, \omega\rangle\right)^k - \sum_{i=1,-1}(1+iX)^k\right\}d\omega \tag{4.1}$$

where $k = s/2 > 1$. To estimate the integrand $\{\cdots\}$ we shall use the following inequality: For all $a \in [-1, 1]$ and $t \in [-1, 1]$

$$(1+at)^k + (1-at)^k - (1+a)^k - (1-a)^k + \frac{k(k-1)}{2}a^2(1-t^2) \le 0\,. \tag{4.2}$$

This inequality is easily proven by checking that the left hand side is a convex function in $t \in [-1, 1]$. Applying (4.2) to $a = X$ we have

$$\sum_{i=1,-1}(1 + iX\cos\theta)^k - \sum_{i=1,-1}(1+iX)^k \le -\frac{k(k-1)}{2}X^2\sin^2\theta \tag{4.3}$$

from which we see that if $Y=0$ then the lemma holds true. Suppose $Y\neq 0$. Then $|\langle \mathbf{h},\mathbf{k}\rangle|<1$. By the Cauchy-Schwarz inequality and $r=\frac{|v+v_*||v-v_*|}{\langle v\rangle^2+\langle v_*\rangle^2}<1$, we have that for all $t\in[0,1]$,

$$1-(|X|\cos\theta\,+t|Y|\sin\theta)>1-\sqrt{\langle \mathbf{h},\mathbf{k}\rangle^2+t^2(1-\langle \mathbf{h},\mathbf{k}\rangle^2)}\geq \frac{1}{2}(1-\langle \mathbf{h},\mathbf{k}\rangle^2)(1-t^2)\,. \tag{4.4}$$

Applying Taylor's formula to the function

$$t\mapsto \sum_{i=1,-1}(1+iX\cos\theta+ti\,Y\sin\theta\,\langle \mathbf{j},\omega\rangle))^k-\sum_{i=1,-1}(1+iX)^k\ ,\quad t\in[0,1]\ ,$$

we compute

$$\begin{aligned}&\Big\{\cdots\Big\}=\sum_{i=1,-1}(1+iX\cos\theta)^k-\sum_{i=1,-1}(1+iX)^k\\ &+k\sum_{i=1,-1}(1+iX\cos\theta)^{k-1}\,iY\sin\theta\,\langle \mathbf{j},\omega\rangle\\ &+k(k-1)\,(Y\sin\theta\,\langle \mathbf{j},\omega\rangle)^2\int_0^1(1-t)\sum_{i=1,-1}\Big(1+iX\cos\theta+tiY\sin\theta\,\langle \mathbf{j},\omega\rangle\Big)^{k-2}dt\,.\end{aligned}$$

Since $\int_{\mathbf{S}^{N-2}(\mathbf{k})}\langle \mathbf{j},\omega\rangle d\omega=0$, it follows from (4.1) and (4.3) that

$$W(v,v_*,\theta)\leq -\frac{k(k-1)}{2}|\mathbf{S}^{N-2}|X^2\sin^2\theta+k(k-1)Y^2\sin^2\theta\int_{\mathbf{S}^{N-2}(\mathbf{k})}Z_k(\omega)d\omega \tag{4.5}$$

where

$$Z_k(\omega)=\int_0^1(1-t)\sum_{i=1,-1}\Big(1+iX\cos\theta+tiY\sin\theta\,\langle \mathbf{j},\omega\rangle\Big)^{k-2}dt\,.$$

By considering $1<k<2$ (for which we use (4.4)) and $k\geq 2$ respectively, we compute for all $k>1$

$$Y^2Z_k(\omega)\leq 2^{k+1}r^2(1-\langle \mathbf{h},\mathbf{k}\rangle^2)^{\bar{k}}\qquad \forall\,\omega\in \mathbf{S}^{N-2}(\mathbf{k}) \tag{4.6}$$

where $\bar{k}=\min\{k-1,1\}$. Since $-X^2\leq r^2\Big((1-\langle \mathbf{h},\mathbf{k}\rangle^2)^{\bar{k}}-1\Big)$, it follows from (4.5) and (4.6) that

$$W(v,v_*,\theta)\leq \frac{k(k-1)}{2}|\mathbf{S}^{N-2}|r^2\sin^2\theta\Big\{2^{k+3}(1-\langle \mathbf{h},\mathbf{k}\rangle^2)^{\bar{k}}-1\Big\}\,.$$

This proves the lemma. $\Box$

**Lemma 4.2.** *Let $B(v-v_*,\sigma)$ satisfy (1.6) and (1.7) with the constants $A^*,A_*$. Let $L[\Delta\langle\cdot\rangle^s](v,v_*)$ be defined in (1.9) for $\varphi(v)=\langle v\rangle^s$. Then for any $s>2$ we have*

(I) *If $-2\leq\gamma<0$, then for any $\varepsilon>0$*

$$\begin{aligned}&L[\Delta\langle\cdot\rangle^s](v,v_*)\\ &\leq -c_s\left(\langle v\rangle^{s+\gamma}+\langle v_*\rangle^{s+\gamma}\right)+\varepsilon C_s\left(\langle v\rangle^{s+\gamma}\langle v_*\rangle^2+\langle v_*\rangle^{s+\gamma}\langle v\rangle^2\right)+C_{s,\varepsilon}\langle v\rangle^2\langle v_*\rangle^2\,.\end{aligned}$$

(II) *If $-4\leq\gamma<-2$, then for any $\lambda\geq 1$*

$$\begin{aligned}&\mathbf{1}_{\{|v-v_*|>\lambda\}}L[\Delta\langle\cdot\rangle^s](v,v_*)\\ &\leq -c_s\left(\langle v\rangle^{s+\gamma}+\langle v_*\rangle^{s+\gamma}\right)+C_s\lambda^{2+\gamma}\left(\langle v\rangle^{s+\gamma}\langle v_*\rangle^2+\langle v_*\rangle^{s+\gamma}\langle v\rangle^2\right)+C_{s,\lambda}\langle v\rangle^2\langle v_*\rangle^2\,.\end{aligned}$$

*Here the constants $0 < c_s, C_s < \infty$ depend only on $N, A^*, A_*$ and $s$, while $0 < C_{s,\varepsilon}, C_{s,\lambda} < \infty$ depend also on $\varepsilon$ and $\lambda$ respectively.*

*Proof.* By symmetry $L[\Delta\langle\cdot\rangle^s](v,v_*) = L[\Delta\langle\cdot\rangle^s](v_*,v)$, we may assume that $|v| \ge |v_*|$. We note also that

$$\langle v\rangle^{s+\gamma} \ge \frac{1}{2}\left(\langle v\rangle^{s+\gamma} + \langle v_*\rangle^{s+\gamma}\right) \qquad \text{if} \quad |v| \ge |v_*| \quad \text{and} \quad s+\gamma \ge 0\,. \tag{4.7}$$

By Lemma 4.1 and $\Delta\langle\cdot\rangle^s = \langle v'\rangle^s + \langle v'_*\rangle^s - \langle v\rangle^s - \langle v_*\rangle^s$, we have

$$\begin{aligned}
&L[\Delta\langle\cdot\rangle^s](v,v_*) \\
&\le s(s-2)\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{(s-4)/2}|v+v_*|^2|v-v_*|^2\left[(1-\langle \mathbf{h},\mathbf{k}\rangle^2)^{\bar{s}/2} - 2^{-s/2-3}\right] \\
&\times \int_{\mathbf{S}^{N-1}} B(v-v_*,\sigma)\sin^2\theta\, d\sigma\,.
\end{aligned}$$

Let $R_s = 2^{(\bar{s}+4+s/2)/\bar{s}}$, and consider $L[\Delta\langle\cdot\rangle^s](v,v_*) = L_1(v,v_*) + L_2(v,v_*)$ where

$$L_1(v,v_*) := L[\Delta\langle\cdot\rangle^s](v,v_*)\mathbf{1}_{\{|v|\le R_s|v_*|\}}\,, \quad L_2(v,v_*) := L[\Delta\langle\cdot\rangle^s](v,v_*)\mathbf{1}_{\{|v|> R_s|v_*|\}}\,.$$

By assumption (1.6) we have

$$L_1(v,v_*) \le C_s\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{(s-4)/2}|v+v_*|^2|v-v_*|^{2+\gamma}\mathbf{1}_{\{|v|\le R_s|v_*|\}}\,. \tag{4.8}$$

To estimate $L_2(v,v_*)$, observing that

$$|v| > R_s|v_*| \implies 1 - \langle \mathbf{h},\mathbf{k}\rangle^2 \le \frac{4|v|^2|v_*|^2}{(|v|^2+|v_*|^2)^2} < \frac{4}{R_s^2}\ ,$$

we have by the choice of $R_s$ that

$$\left((1-\langle \mathbf{h},\mathbf{k}\rangle^2)^{\bar{s}/2} - 2^{-3-s/2}\right)\mathbf{1}_{\{|v|> R_s|v_*|\}} \le -2^{-4-s/2}\mathbf{1}_{\{|v|> R_s|v_*|\}}\,.$$

Thus, using the assumption (1.7), we obtain

$$L_2(v,v_*) \le -c_s\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{(s-4)/2}|v+v_*|^2|v-v_*|^2(1+|v-v_*|)^{\gamma}\mathbf{1}_{\{|v|> R_s|v_*|\}}\,. \tag{4.9}$$

(I) Assume $-2 \le \gamma < 0$. Using $|v \pm v_*| \le 2^{1/2}\left(\langle v\rangle^2 + \langle v_*\rangle^2\right)^{1/2}$ and recalling $s > 2$ and $|v| \ge |v_*|$, we have by (4.8) that

$$L_1(v,v_*) \le C_s\langle v\rangle^{s+\gamma}\mathbf{1}_{\{|v|\le R_s|v_*|\}} \le C_s\langle v\rangle^{s+\gamma-2}\langle v_*\rangle^2\,.$$

Applying the elementary inequality

$$a^{k-2}b^2 \le \varepsilon a^k b^2 + (1+\varepsilon^{-(k-2)/2})a^2b^2\,, \qquad a,b \ge 1\,,\ k \in \mathbf{R}^1\,,\ \varepsilon > 0\ , \tag{4.10}$$

we get for any $\varepsilon > 0$

$$L_1(v,v_*) \le \varepsilon C_s\langle v\rangle^{s+\gamma}\langle v_*\rangle^2 + C_{s,\varepsilon}\langle v\rangle^2\langle v_*\rangle^2\,.$$

To estimate $L_2(v,v_*)$, we observe by $R_s > 2$ and $\gamma < 0$ that

$$|v| > R_s|v_*| \ \text{and}\ |v| > 1 \implies |v \pm v_*| \ge \frac{1}{4}\langle v\rangle \ \text{and}\ (1+|v-v_*|)^{\gamma} \ge 4^{\gamma}\langle v\rangle^{\gamma}\,.$$

By (4.9) (considering $s \le 4$ and $s > 4$ respectively) we have

$$\begin{aligned}
L_2(v, v_*) &\le L_2(v, v_*)\mathbf{1}_{\{|v|>1\}} \le -c_s\langle v\rangle^{s+\gamma}\mathbf{1}_{\{|v|>R_s|v_*|\}}\mathbf{1}_{\{|v|>1\}} \\
&= c_s\langle v\rangle^{s+\gamma}\Big( -1 + \mathbf{1}_{\{|v|>R_s|v_*|\}}\mathbf{1}_{\{|v|\le 1\}} + \mathbf{1}_{\{|v|\le R_s|v_*|\}}\Big) .
\end{aligned}$$

Since $\langle v\rangle^{s+\gamma}\mathbf{1}_{\{|v|>R_s|v_*|\}}\mathbf{1}_{\{|v|\le 1\}} \le C_s$, and by (4.8) for any $\varepsilon > 0$,

$$\langle v\rangle^{s+\gamma}\mathbf{1}_{\{|v|\le R_s|v_*|\}} \le C_s\langle v\rangle^{s+\gamma-2}\langle v_*\rangle^2 \le \varepsilon C_s\langle v\rangle^{s+\gamma}\langle v_*\rangle^2 + C_{s,\varepsilon}\langle v\rangle^2\langle v_*\rangle^2 \ ,$$

it follows that

$$L_2(v, v_*) \le -c_s\langle v\rangle^{s+\gamma} + \varepsilon C_s\langle v\rangle^{s+\gamma}\langle v_*\rangle^2 + C_{s,\varepsilon}\langle v\rangle^2\langle v_*\rangle^2 .$$

Therefore,

$$L[\Delta\langle\cdot\rangle^s](v, v_*) = L_1(v, v_*) + L_2(v, v_*) \le -c_s\langle v\rangle^{s+\gamma} + \varepsilon C_s\langle v\rangle^{s+\gamma}\langle v_*\rangle^2 + C_{s,\varepsilon}\langle v\rangle^2\langle v_*\rangle^2 .$$

This together with (4.7) (because $s + \gamma > 0$) gives the inequality in part (I) of the lemma.

(II) Assume $-4 \le \gamma < -2$. Given any $\lambda \ge 1$. By (4.8) and $2 + \gamma < 0$ we have

$$\mathbf{1}_{\{|v-v_*|>\lambda\}}L_1(v, v_*) \le C_s\lambda^{2+\gamma}\mathbf{1}_{\{|v|\le R_s|v_*|\}}\langle v\rangle^{s-2} \le C_s\lambda^{2+\gamma}\langle v\rangle^{s+\gamma}\langle v_*\rangle^2 . \tag{4.11}$$

Note that if $s \le 4$, then $s + \gamma \le 2$ so that $\langle v\rangle^{s+\gamma} \le \langle v\rangle^2$ and $\langle v\rangle^{s+\gamma}, \langle v_*\rangle^{s+\gamma} \le \langle v\rangle^2\langle v_*\rangle^2$ and thus by (4.8) and neglecting the non-positive term $\mathbf{1}_{\{|v-v_*|>\lambda\}}L_2(v, v_*)$ we get

$$\mathbf{1}_{\{|v-v_*|>\lambda\}}L[\Delta\langle\cdot\rangle^s](v, v_*) \le C_s\lambda^{2+\gamma}\langle v\rangle^2\langle v_*\rangle^2 \le -(\langle v\rangle^{s+\gamma} + \langle v_*\rangle^{s+\gamma}) + C_{s,\lambda}\langle v\rangle^2\langle v_*\rangle^2 \ ,$$

which is a special case of the inequality in part (II) of the lemma. Next, assume $s > 4$. To estimate $\mathbf{1}_{\{|v-v_*|>\lambda\}}L_2(v, v_*)$ we see from $R_s > 2$ and $\lambda \ge 1$ that

$$|v - v_*| > \lambda \ \text{and} \ |v| > R_s|v_*| \implies |v \pm v_*| \ge \frac{1}{2}|v| \,, \ 1 < |v - v_*| \le 2|v| \,, \ \text{and} \ |v| \ge \frac{1}{4}\langle v\rangle .$$

This implies by (4.9) and $\gamma < 0$ that

$$\begin{aligned}
&\mathbf{1}_{\{|v-v_*|>\lambda\}}L_2(v, v_*) \le -c_s\, \mathbf{1}_{\{|v-v_*|>\lambda\}}\mathbf{1}_{\{|v|>R_s|v_*|\}}\langle v\rangle^{s+\gamma} \\
&= c_s\left(-1 + \mathbf{1}_{\{|v|\le R_s|v_*|\}} + \mathbf{1}_{\{|v-v_*|\le\lambda\}}\mathbf{1}_{\{|v|>R_s|v_*|\}}\right)\langle v\rangle^{s+\gamma} .
\end{aligned}$$

Since $R_s > 2$ and $|v| \le R_s|v_*|$ imply $\langle v\rangle \le R_s\langle v_*\rangle$, it follows from the inequality (4.10) that for any $\varepsilon > 0$,

$$\mathbf{1}_{\{|v|\le R_s|v_*|\}}\langle v\rangle^{s+\gamma} \le \varepsilon C_s\langle v\rangle^{s+\gamma}\langle v_*\rangle^2 + C_{s,\varepsilon}\langle v\rangle^2\langle v_*\rangle^2 .$$

Also, we see that $|v - v_*| \le \lambda$ and $|v| > R_s|v_*| \Longrightarrow |v| \le 2\lambda$. Thus,

$$\mathbf{1}_{\{|v-v_*|\le\lambda\}}\mathbf{1}_{\{|v|>R_s|v_*|\}}\langle v\rangle^{s+\gamma} \le C_{s,\lambda} \le C_{s,\lambda}\langle v\rangle^2\langle v_*\rangle^2 .$$

Let us now choose $\varepsilon = \lambda^{2+\gamma}$. Then

$$\mathbf{1}_{\{|v-v_*|>\lambda\}}L_2(v, v_*) \le -c_s\langle v\rangle^{s+\gamma} + C_s\lambda^{2+\gamma}\langle v\rangle^{s+\gamma}\langle v_*\rangle^2 + C_{s,\lambda}\langle v\rangle^2\langle v_*\rangle^2 . \tag{4.12}$$

Summarizing (4.11) and (4.12) gives

$$\mathbf{1}_{\{|v-v_*|>\lambda\}}L[\Delta\langle\cdot\rangle^s](v,v_*)\le -c_s\langle v\rangle^{s+\gamma}+C_s\lambda^{2+\gamma}\langle v\rangle^{s+\gamma}\langle v_*\rangle^2+C_{s,\lambda}\langle v\rangle^2\langle v_*\rangle^2$$

which together with $s+\gamma>0$ and (4.7) gives the inequality in part (II) of the lemma. □

**Proof of Theorem 1: Moment Estimates**. We use a short notation

$$\|f\|_s:=\|f\|_{L^1_s}\,.$$

Let $f(v,t)$ be a weak solution of Eq.(B) with $f|_{t=0}=f_0\in L^1_{(1,0,1)}\cap L^1_s\cap L^1\log L(\mathbf{R}^N)$ and $s>2$. Recall that $f$ conserves the mass and energy: $\|f(t)\|_0\equiv 1\,,\ \|f(t)\|_2\equiv 1+N$.

**Step 1**. We shall prove that $f\in L^\infty_{loc}([0,\infty),L^1_s(\mathbf{R}^N))$, i.e.

$$\sup_{t\in[0,T]}\|f(t)\|_s<\infty\quad \forall\, T<\infty\,. \tag{4.13}$$

For any $\lambda\ge 1$, we split $B=B_\lambda+B^\lambda$ as in (2.11) and let $L_\lambda[\cdot]$, $Q^\lambda(\cdot)$ correspond to $B_\lambda(z,\sigma),B^\lambda(z,\sigma)$ respectively. Let $\chi\in C^\infty_c(\mathbf{R}^N)$ be the function used above (see (3.1)). For any $k\ge 2$, $n\ge 1$, let

$$\varphi_k(v)=\langle v\rangle^k\,,\quad \varphi_{k,n}(v)=\varphi_k(v)\chi(v/n)\,.$$

Then $\varphi_{k,n}\in C^2_b(\mathbf{R}^N)$ and so for $-2\le\gamma<0$

$$\int_{\mathbf{R}^N}\varphi_{k,n}(v)f(v,t)dv=\int_{\mathbf{R}^N}\varphi_{k,n}(v)f_0(v)dv+\frac{1}{2}\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}L[\Delta\varphi_{k,n}]ff_*dv_*dv \tag{4.14}$$

and for $-4\le\gamma<-2$

$$\begin{aligned}\int_{\mathbf{R}^N}\varphi_{k,n}(v)f(v,t)dv&=\int_{\mathbf{R}^N}\varphi_{k,n}(v)f_0(v)dv\\&\quad-\frac{1}{4}\int_0^t d\tau\int_{\mathbf{R}^N}Q^\lambda(f\,|\Delta\varphi_{k,n})dv+\frac{1}{2}\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}L_\lambda[\Delta\varphi_{k,n}]ff_*dv_*dv\end{aligned} \tag{4.15}$$

where we used (2.12). Moreover

$$0\le\varphi_{k,n}(v)\le\varphi_k(v)\,,\quad \lim_{n\to\infty}\varphi_{k,n}(v)=\varphi_k(v)\,,$$

$$\lim_{n\to\infty}\Delta\varphi_{k,n}(v',v'_*,v,v_*)=\Delta\varphi_k(v',v'_*,v,v_*)\,,$$

$$\lim_{n\to\infty}L[\Delta\varphi_{k,n}](v,v_*)=L[\Delta\varphi_k](v,v_*)\,,\quad \lim_{n\to\infty}L_\lambda[\Delta\varphi_{k,n}](v,v_*)=L_\lambda[\Delta\varphi_k](v,v_*)\,,$$

$$|\partial^2\varphi_k(v)|\,,\ \sup_{n\ge1}|\partial^2\varphi_{k,n}(v)|\le C_k\langle v\rangle^{k-2}\,,$$

and by introducing

$$\Psi_{k,\alpha}(v,v_*):=(1+|v|^2+|v_*|^2)^{(k-2)/2}|v-v_*|^\alpha$$

and using Lemma 2.1, and (1.9), we have

$$|\Delta\varphi_k|\,,\quad \sup_{n\ge1}|\Delta\varphi_{k,n}|\le C_k\Psi_{k,2}(v,v_*)\sin\theta \tag{4.16}$$

$$\left|\int_{\mathbf{S}^{N-2}(\mathbf{k})}\Delta\varphi_k d\omega\right|,\ \sup_{n\ge 1}\left|\int_{\mathbf{S}^{N-2}(\mathbf{k})}\Delta\varphi_{k,n}d\omega\right|\le C_k\Psi_{k,2}(v,v_*)\sin^2\theta \tag{4.17}$$

$$|L[\Delta\varphi_k](v,v_*)|\,,\ \sup_{n\ge1}|L[\Delta\varphi_{k,n}](v,v_*)|\le C_k\Psi_{k,2+\gamma}(v,v_*)\,. \tag{4.18}$$

The constants $C_k$ depend only on $k,N,\gamma,A^*$. In the following we assume $2\le k\le s$.

Suppose $-2\le\gamma<0$. In (4.14) letting $n\to\infty$ we obtain by Fatou's Lemma and (4.18) that

$$\|f(t)\|_{L^1_k}\le\|f_0\|_s+C_k\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}\Psi_{k,2+\gamma}ff_*dv_*dv\,.$$

Since $-2\le\gamma<0$ implies

$$\Psi_{k,2+\gamma}(v,v_*)\le C_k\left(\langle v\rangle^{k+\gamma}\langle v_*\rangle^2+\langle v_*\rangle^{k+\gamma}\langle v\rangle^2\right)$$

it follows form the conservation of mass and energy that

$$\|f(t)\|_k\le\|f_0\|_s+C_k\int_0^t\|f(\tau)\|_{k+\gamma}d\tau\,,\quad t\ge0\,.$$

Now take $k=k_m=\min\{s,2+m|\gamma|\},m=1,2,...,$. By $\gamma<0$ and induction on $m$ we then obtain

$$\|f(t)\|_{k_m}\le C_{k_m}(1+t)^m\,,\quad t\ge0\,,\quad m=1,2,\dots\,.$$

This proves (4.13) for $-2\le\gamma<0$. Next suppose $-4\le\gamma<-2$. Observing that $|v-v_*|\le\lambda\Longrightarrow 1+|v|^2+|v_*|^2\le4\lambda\langle v\rangle\langle v_*\rangle$ and $|v-v_*|^{4+\gamma}\le\lambda^{4+\gamma}$, we have

$$\mathbf{1}_{\{|v-v_*|\le\lambda\}}[\Psi_{k,2}(v,v_*)]^2|v-v_*|^\gamma\le C_k\lambda^{k+2+\gamma}\langle v\rangle^{k-2}\langle v_*\rangle^{k-2}\,,$$

$$\int_{\mathbf{R}^N\times\mathbf{R}^N}\mathbf{1}_{\{|v-v_*|\le\lambda\}}[\Psi_{k,2}(v,v_*)]^2|v-v_*|^\gamma ff_*dv_*dv\le C_k\lambda^{k+2+\gamma}\|f(\tau)\|^2_{k-2}\,.$$

Therefore using (4.16) and Lemma 2.2 we obtain

$$\left|\int_0^t d\tau\int_{\mathbf{R}^N}Q^\lambda(f\,|\,\Delta\varphi_{k,n})dv\right|\le C_k\lambda^{(k+2+\gamma)/2}\int_0^t\|f(\tau)\|_{k-2}\sqrt{D(f(\tau))}\,d\tau\,. \tag{4.19}$$

Next, by $2+\gamma<0$ and $\lambda\ge1$ we have

$$\mathbf{1}_{\{|v-v_*|>\lambda\}}\Psi_{k,2+\gamma}(v,v_*)\le C_k\left(\langle v\rangle^{k-2}+\langle v_*\rangle^{k-2}\right)$$

which gives by (4.18) and $L_\lambda[\cdot]=\mathbf{1}_{\{|v-v_*|>\lambda\}}L[\cdot]$ that

$$\frac12\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}L_\lambda[\Delta\varphi_{k,n}]ff_*dv_*dv\le C_k\int_0^t\|f(\tau)\|_{k-2}\,d\tau\,. \tag{4.20}$$

In the equation (4.15), letting $\lambda=1$ and $n\to\infty$, we obtain from (4.19), (4.20) and Fatou's Lemma that

$$\|f(t)\|_k\le\|f_0\|_s+C_k\int_0^t\|f(\tau)\|_{k-2}\left(1+\sqrt{D(f(\tau))}\right)\,d\tau\,.$$

Now we choose $k=k_m=\min\{s,2m\}$. Then by induction on $m$ it is easy to show that there are constants $0<C_{k_m}<\infty$ such that

$$\|f(t)\|_{k_m}\le C_{k_m}(1+t)^{m-1}\,,\quad t\ge0\,,\quad m=1,2,\dots\,.$$

This proves (4.13) for $-4 \le \gamma < -2$.

**Step 2**. We shall prove the following inequality

$$\|f(t)\|_s + c_s \int_0^t \|f(\tau)\|_{s+\gamma} d\tau \le C_s(1+t)\,, \quad t \ge 0 \tag{4.21}$$

which implies (1.16) because $\gamma < 0$. Here and below the constants $0 < c_s\,, C_s < \infty$ depend only on $N, \gamma, A_*, A^*, s$ and $\|f_0\|_{L^1_s}$, and in case $-4 \le \gamma < -2$, they depend also on $H(f_0)$.

¿From the pointwise bounds (4.16)-(4.18) and integrability shown in Step 1 we see that the dominated convergence theorem can be used and we get from (4.14) and (4.15) with $k = s$ that for $-2 \le \gamma < 0$

$$\|f(t)\|_s = \|f_0\|_s + \frac{1}{2}\int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N} L[\Delta\varphi_s] f f_* dv_* dv\,, \tag{4.22}$$

and for $-4 \le \gamma < -2$

$$\|f(t)\|_s = \|f_0\|_s - \frac{1}{4}\int_0^t d\tau \int_{\mathbf{R}^N} Q^\lambda(f\,|\,\Delta\varphi_s) dv + \frac{1}{2}\int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N} L_\lambda[\Delta\varphi_s] f f_* dv_* dv\,. \tag{4.23}$$

To prove (4.21), we first consider $-2 \le \gamma < 0$. By Lemma 4.2 (recalling $\varphi_s(v) = \langle v\rangle^s$) and the conservation of mass and energy we have from (4.22) that for any $\varepsilon > 0$

$$\|f(t)\|_s \le \|f_0\|_s - (c_s - \varepsilon C_s)\int_0^t \|f(\tau)\|_{s+\gamma} d\tau + C_{s,\varepsilon} t\,, \quad t \ge 0\,.$$

Therefore choosing $\varepsilon = \frac{c_s}{2C_s}$ leads to (4.21) (with different constants).

Next assume $-4 \le \gamma < -2$. Using inequality (4.19) for $k = s$ and letting $n \to \infty$ we have

$$\left|\int_0^t d\tau \int_{\mathbf{R}^N} Q^\lambda(f\,|\Delta\varphi_s) dv\right| \le C_s \lambda^{(s+2+\gamma)/2}\int_0^t \|f(\tau)\|_{s-2}\sqrt{D(f(\tau))}\,d\tau\,.$$

By the Cauchy-Schwarz inequality and using $s - 4 \le s + \gamma$,

$$\|f(\tau)\|_{s-2} \le \sqrt{\|f(\tau)\|_s}\sqrt{\|f(\tau)\|_{s-4}} \le \sqrt{\|f(\tau)\|_s}\sqrt{\|f(\tau)\|_{s+\gamma}}\,.$$

Therefore, for any $\varepsilon > 0$

$$\lambda^{(s+2+\gamma)/2}\|f(\tau)\|_{s-2}\sqrt{D(f(\tau))} \le \frac{1}{2}\varepsilon\|f(\tau)\|_{s+\gamma} + \frac{1}{2\varepsilon}\lambda^{s+2+\gamma}\|f(\tau)\|_s D(f(\tau))\,,$$

and so

$$\frac{1}{4}\left|\int_0^t d\tau \int_{\mathbf{R}^N} Q^\lambda(f\,|\,\Delta\varphi_s) dv\right| \le \varepsilon C_s \int_0^t \|f(\tau)\|_{s+\gamma} d\tau + C_{s,\varepsilon,\lambda}\int_0^t \|f(\tau)\|_s D(f(\tau)) d\tau\,.$$

Also, using Lemma 4.2 we have as shown above that

$$\frac{1}{2}\int_0^t d\tau \int_{\mathbf{R}^N\times\mathbf{R}^N} L_\lambda[\Delta\varphi_s] f f_* dv_* dv \le -\left(c_s - C_s\lambda^{2+\gamma}\right)\int_0^t \|f(\tau)\|_{s+\gamma} d\tau + C_{s,\lambda}\, t\,.$$

Choose $\varepsilon = \lambda^{2+\gamma}$. Then from (4.23) and the above estimates we obtain

$$\|f(t)\|_s \le \|f_0\|_s - \left(c_s - C_s\lambda^{2+\gamma}\right)\int_0^t \|f(\tau)\|_{s+\gamma} d\tau + C_{s,\lambda}\int_0^t \|f(\tau)\|_s D(f(\tau)) d\tau + C_{s,\lambda} t\,.$$

We can assume that $C_s \ge c_s$. By $2+\gamma<0$ we can choose

$$\lambda = \left(\frac{c_s}{2C_s}\right)^{1/(2+\gamma)} \qquad (>1).$$

Then with different constants we obtain

$$\|f(t)\|_s + c_s\int_0^t \|f(\tau)\|_{s+\gamma}d\tau \le C_s(1+t) + C_s\int_0^t \|f(\tau)\|_s D(f(\tau))d\tau\,, \quad t\ge 0\,.$$

By Gronwall's Lemma, this gives

$$\|f(t)\|_s + c_s\int_0^t \|f(\tau)\|_{s+\gamma}d\tau \le C_s(1+t)\exp(C_s\int_0^t D(f(\tau))d\tau)\,, \quad t\ge 0$$

which implies (4.21) because $\int_0^\infty D(f(\tau))d\tau \le C_{N,H(f_0)} < \infty$. $\square$

# 5 Convergence to Equilibrium in Broad Generality

In this section we give a unified treatment of strong convergence to equilibrium which includes all cases (soft potentials, hard potentials, etc., with and without cutoff). In the following theorem, the functions $f(v,t)$ are not even assumed to be solutions of the Boltzmann equation: Their only connection to the Boltzmann equation is the requirement (5.3) below in the integrated entropy dissipation.

**Theorem 4.** *Let $B(z,\sigma)$ be a collision kernel satisfying $B(z,\sigma)>0$ a.e. $(z,\sigma)\in \mathbf{R}^N\times\mathbf{S}^{N-1}$, and let $0\le f\in L^\infty([0,\infty); L^1_2(\mathbf{R}^N))$ satisfy*

$$f(\cdot,t)\in L^1_{(1,0,1)}(\mathbf{R}^N) \quad \forall\, t\ge 0\,; \quad \sup_{t\ge 0}\int_{\mathbf{R}^N} f(v,t)\Phi(f(v,t))dv < \infty \tag{5.1}$$

$$\lim_{|t_1-t_2|\to 0}\int_{\mathbf{R}^N}\varphi(v)\Big(f(v,t_1)-f(v,t_2)\Big)dv = 0 \quad \forall\,\varphi\in C_c^\infty(\mathbf{R}^N)\,, \tag{5.2}$$

$$\int_0^\infty D(f(t))dt < \infty\,, \tag{5.3}$$

$$\|f(t)\|_{L^1_{2+}} := \int_{\mathbf{R}^N}\langle v\rangle^2\Psi(v)f(v,t)dv < \infty \quad \forall\, t\ge t_0 \tag{5.4}$$

*for some $t_0\ge 0$, where $\Phi(r)\ge 0, \Psi(v)\ge 1$ satisfy $\lim\limits_{r\to\infty}\Phi(r)=\infty$, $\lim\limits_{|v|\to\infty}\Psi(v)=\infty$, and $D(f)$ is defined in (1.8) with the present $B(z,\sigma)$. Then for the Maxwellian $M\in L^1_{(1,0,1)}(\mathbf{R}^N)$ given by (1.14) we have*

*(I) If*

$$\sup_{T\ge t_0}\frac{1}{T}\int_{t_0}^T \|f(t)\|_{L^1_{2+}}dt < \infty\,, \tag{5.5}$$

*then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \|f(t)-M\|_{L^1_2}dt = 0\,. \tag{5.6}$$

*(II) If*

$$\lim_{|t_1-t_2|\to 0}\|f(t_1)-f(t_2)\|_{L^1}=0\,,\quad \sup_{t\ge t_0}\|f(t)\|_{L^1_{2+}}<\infty\,, \tag{5.7}$$

*then*

$$\lim_{t\to\infty}\|f(t)-M\|_{L^1_2}=0\,. \tag{5.8}$$

*Proof.* First of all, we use $\|f(t)-M\|_{L^1_2}\le C_N\sqrt{\|f(t)-M\|_{L^1}}$ (see (1.15)). Hence, to prove the theorem, it suffices to prove the apparently weaker result in which $\|f(t)-M\|_{L^1_2}$ is replaced by $\|f(t)-M\|_{L^1}$. Also, if we let $D_{\min}(f)$ correspond to $B_{\min}(z,\sigma):=\min\{B(z,\sigma)\,,1\}$, then $D_{\min}(f(t))\le D(f(t))$, so that by replacing $B(z,\sigma)$ with $B_{\min}(z,\sigma)$, we can assume for notational convenience that $B$ is bounded: $B(z,\sigma)\le 1$.

¿From (5.1) we have

$$\sup_{t\ge 0}\int_{\mathbf{R}^N}f(v,t)\left(\langle v\rangle^2+\Phi(f(v,t))\right)dv<\infty \tag{5.9}$$

which implies that $\{f(\cdot,t)\}_{t\ge 0}$ is weakly compact in $L^1(\mathbf{R}^N)$. Here and below "compact in $L^1$" always means "relatively compact in $L^1$". We now split $Q(f)$ as usual as

$$Q(f)(v,t)=Q^+(f)(v,t)-Q^-(f)(v,t) \tag{5.10}$$

where

$$Q^+(f)(v,t)=\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}}B(v-v_*,\sigma)f'f'_*d\sigma dv_*\,, \tag{5.11}$$

$$Q^-(f)(v,t)=\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}}B(v-v_*,\sigma)ff_*d\sigma dv_*=f(v,t)L(f)(v,t) \tag{5.12}$$

$$L(f)(v,t)=\int_{\mathbf{R}^N}\|B(z,\cdot)\|_{L^1(\mathbf{S}^{N-1})}f(v-z,t)dz\,. \tag{5.13}$$

Then using the special identity $Q^+(M)(v)=M(v)L(M)(v)$ we have

$$(f-M)L(M)=-Q(f)+Q^+(f)-Q^+(M)-fL(f-M)\,. \tag{5.14}$$

As shown in the proof of Lemma 2.2, applying the inequality (2.10) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|Q(f)(t)\|_{L^1}&\le\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}}B(v-v_*,\sigma)|f'f'_*-ff_*|d\sigma dv_*dv\\ &\le\sqrt{4\int_{\mathbf{R}^N}f(v,t)L(f)(v,t)dv}\sqrt{D(f(t))}\le\sqrt{4|\mathbf{S}^{N-1}|}\sqrt{D(f(t))}\,.\end{aligned} \tag{5.15}$$

Let us write for any $0<R<\infty$

$$f-M=\mathbf{1}_{\{|v|\le R\}}\frac{1}{L(M)}(f-M)L(M)+\mathbf{1}_{\{|v|>R\}}(f-M)\,.$$

Then using (5.14), (5.15) we have

$$\|f(t)-M\|_{L^1}\le\frac{1}{L_R}\Big(C_N\sqrt{D(f(t))}+E(t)\Big)+\frac{2N}{R^2}\,,\quad t\ge 0 \tag{5.16}$$

where $L_R = \min_{|v|\le R} L(M)(v) > 0$ and

$$E(t) = \|Q^+(f)(t) - Q^+(M)\|_{L^1} + \|f(t)g(t)\|_{L^1}\,, \quad g(v,t) = L(f-M)(v,t)\,.$$

Here the positivity $L_R > 0$ is obvious because the function $v \mapsto L(M)(v)$ is continuous on $\mathbf{R}^N$ and, by the assumption on $B$, $\|B(z,\cdot)\|_{L^1(\mathbf{S}^{N-1})} > 0$ a.e. $z \in \mathbf{R}^N$.

We next prove that for any sequence $\{t_n\}_{n\ge1} \subset [t_0,\infty)$ satisfying $t_n \to \infty\,(n\to\infty)$ and

$$\sup_{n\ge1} \|f(t_n)\|_{L^1_{2+}} < \infty\ , \tag{5.17}$$

there exists a subsequence, still denoted by $\{t_n\}_{n\ge1}$, and a sequence $\{\bar t_n\}_{n\ge1}$ satisfying $0 \le \bar t_n - t_n \to 0\,(n\to\infty)$, such that

$$\lim_{n\to\infty} E(t_n) = \lim_{n\to\infty} E(\bar t_n) = 0\,, \quad \lim_{n\to\infty} D(f(\bar t_n)) = 0\,. \tag{5.18}$$

To do this we use the fact that

$$\delta_n := \sqrt{\int_{t_n}^{\infty} D(f(t))dt + \frac{1}{n}} \quad \Longrightarrow \quad \frac{1}{\delta_n}\int_{t_n}^{t_n+\delta_n} D(f(t))dt < \delta_n$$

so that there exist $\bar t_n \in [t_n\,,\, t_n+\delta_n]$ such that $D(f(\bar t_n)) \le \delta_n \to 0\,(n\to\infty)$ by the assumption (5.3). By $L^1$-weak compactness of $\{f(\cdot,t)\}_{t\ge0}$ there exists a subsequence of $\{(t_n,\bar t_n)\}_{n\ge1}$, still denoted by $\{(t_n,\bar t_n)\}_{n\ge1}$, and functions $0 \le f_\infty, \bar f_\infty \in L^1(\mathbf{R}^N)$ such that $f(\cdot,t_n) \to f_\infty$, $f(\cdot,\bar t_n) \to \bar f_\infty\,(n\to\infty)$ weakly in $L^1(\mathbf{R}^N)$. Since $0 \le \bar t_n - t_n \le \delta_n \to 0$, it follows from (5.2) that $f_\infty = \bar f_\infty$ a.e. on $\mathbf{R}^N$. On the other hand, by $f(\cdot,t_n) \in L^1_{(1,0,1)}(\mathbf{R}^N)\,(\forall\, n)$ and (5.17) we see that $f_\infty \in L^1_{(1,0,1)}(\mathbf{R}^N)$. And by convexity and nonnegativity of $(x,y) \mapsto (x-y)\log(x/y)$ and Fatou's Lemma, we obtain

$$D(f_\infty) \le \lim_{n\to\infty} D(f(\bar t_n)) = 0\,.$$

Since $B(z,\sigma) > 0$ for a.e. $(z,\sigma) \in \mathbf{R}^N \times \mathbf{S}^{N-1}$, it follows from the well-known result that $f_\infty(v) = ae^{-b|v-u_0|^2}$ a.e. on $\mathbf{R}^N$ for some constants $a>0\,,b>0$ and $u_0 \in \mathbf{R}^N$. Since $f_\infty, M \in L^1_{(1,0,1)}(\mathbf{R}^N)$, this implies $f_\infty = M$ a.e. on $\mathbf{R}^N$. We therefore conclude that $f(\cdot,t_n) \rightharpoonup M\,, f(\cdot,\bar t_n) \rightharpoonup M\ (n\to\infty)$ weakly in $L^1(\mathbf{R}^N)$.

Next, since $B(z,\sigma)$ is bounded and $f(v,t)$ satisfies (5.9), it follows from Lions' compactness result ([14], see also e.g. [3, 16, 24, 25]) that the set $\{Q^+(f)(\cdot,t)\}_{t\ge0}$ is strongly compact in $L^1(\mathbf{R}^N)$. For convenience of the reader, we give here a short proof. By the criterion of $L^1$-strong compactness and the $L^1_2$- bound $\sup_{t\ge0}\|Q^+(f)(\cdot,t)\|_{L^1_2} \le C_N$ we need only to show that

$$\sup_{t\ge0}\|Q^+(f)(\cdot+h,t) - Q^+(f)(\cdot,t)\|_{L^1} \to 0 \quad \text{as} \quad h \to 0\,. \tag{5.19}$$

To do this, we use truncation

$$1 < R < \infty\,, \quad f_R(v,t) := f(v,t)\mathbf{1}_{\{|v|\le R\}\cap\{f(v,t)\le R\}}\,.$$

Then $v \mapsto Q^+(f_R)(v,t)$ is bounded and compactly supported uniformly in $t$:

$$Q^+(f_R)(v,t) \le R^2\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} \mathbf{1}_{\{|v'|\le R\}\cap\{|v'_*|\le R\}}d\sigma dv_* \le C_R\mathbf{1}_{\{|v|\le\sqrt{2}\,R\}}\,. \tag{5.20}$$

From e.g. [16] there is a strictly positive measurable function $\Psi_B(\xi)$ constructed from $B(z,\sigma)$ satisfying $\Psi_B(\xi)\to\infty$ ($|\xi|\to\infty$) such that

$$\int_{\mathbf{R}^N}\Psi_B(\xi)|\mathcal{F}(Q^+(f_R)(\cdot,t))(\xi)|^2d\xi$$
$$\le C_N\int_{\mathbf{R}^N\times\mathbf{R}^N}f_R^2(v,t)f_R^2(v_*,t)(1+|v-v_*|^2)^N dvdv_*\le C_R\,. \tag{5.21}$$

Here and below $C_R<\infty$ depends only on $N$ and $R$, and $\mathcal{F}(g)(\xi)$ is the Fourier transform:

$$\mathcal{F}(g)(\xi)=\int_{\mathbf{R}^N}g(v)e^{-\mathrm{i}\xi\cdot v}dv\,.$$

From (5.20) we have

$$Q^+(f_R)(v+h,t)-Q^+(f_R)(v,t)=\big(Q^+(f_R)(v+h,t)-Q^+(f_R)(v,t)\big)\,\mathbf{1}_{\{|v|\le 1+\sqrt{2}\,R\}}$$

for all $v,h\in\mathbf{R}^N$ with $|h|\le 1$. Therefore by Cauchy-Schwarz inequality, Parseval identity, and (5.21) (considering $|\xi|\le|h|^{-1/2}$ and $|\xi|>|h|^{-1/2}$ ) we obtain

$$\begin{aligned}&\|Q^+(f_R)(\cdot+h,t)-Q^+(f_R)(\cdot,t)\|_{L^1}\\&\le C_R\left(\int_{\mathbf{R}^N}|1-e^{-\mathrm{i}\xi\cdot h}|^2|\mathcal{F}(Q^+(f_R)(\cdot,t))(\xi)|^2d\xi\right)^{1/2}\\&\le C_R\left(|h|+\sup_{|\xi|>|h|^{-1/2}}\frac{1}{\Psi_B(\xi)}\right)^{1/2}=:C_R\Lambda(h)\qquad\forall\,0<|h|\le 1\,.\end{aligned}$$

On the other hand, there is $R_0>0$ such that for all $R>R_0$ we have $\Phi(R)>0$ and

$$\|Q^+(f_R)(\cdot,t)-Q^+(f)(\cdot,t)\|_{L^1}\le C_N\int_{\{f(v,t)>R\}\cup\{|v|>R\}}f(v,t)dv\le C_N\left(\frac{1}{\Phi(R)}+\frac{1}{R^2}\right)\,.$$

Combining these we obtain for all $0<|h|\le 1$ and all $R>R_0$

$$\sup_{t\ge 0}\|Q^+(f)(\cdot+h,t)-Q^+(f)(\cdot,t)\|_{L^1}\le C_R\Lambda(h)+C_N\left(\frac{1}{\Phi(R)}+\frac{1}{R^2}\right)$$

which implies (5.19) by first letting $h\to 0$ and then letting $R\to\infty$.

For any $\psi\in L^\infty(\mathbf{R}^N)$, the function

$$\Psi(v,v_*):=\int_{\mathbf{S}^{N-1}}B(v-v_*,\sigma)\psi(v')d\sigma$$

belongs to $L^\infty(\mathbf{R}^N\times\mathbf{R}^N)$. By $f(\cdot,t_n)-M\rightharpoonup 0$ weakly in $L^1(\mathbf{R}^N)$ and (5.9) we have

$$\begin{aligned}&\int_{\mathbf{R}^N}\psi(v)\Big(Q^+(f)(v,t_n)-Q^+(M)(v)\Big)dv\\&=\int_{\mathbf{R}^N\times\mathbf{R}^N}\Psi(v,v_*)\Big(f(v,t_n)f(v_*,t_n)-M(v)M(v_*)\Big)dv_*dv\to 0\quad(n\to\infty)\,.\end{aligned}$$

This implies $\lim_{n\to\infty} \|Q^+(f)(t_n) - Q^+(M)\|_{L^1} = 0$ because $\{Q^+(f)(t_n)\}_{n\ge1}$ is strongly compact in $L^1(\mathbf{R}^N)$. Also by weak convergence and $\|B(z,\cdot)\|_{L^1(\mathbf{S}^{N-1})} \le |\mathbf{S}^{N-1}|$ we have

$$g(v,t_n) = \int_{\mathbf{R}^N} \|B(v-v_*,\cdot)\|_{L^1(\mathbf{S}^{N-1})}\Big(f(v_*,t_n) - M(v_*)\Big)dv_* \to 0 \quad (n\to\infty)$$

for all $v\in\mathbf{R}^N$. Since $|g(v,t_n)| \le |\mathbf{S}^{N-1}|$, it follows from (5.9) that $\lim_{n\to\infty}\|f(t_n)g(t_n)\|_{L^1} = 0$. Thus $\lim_{n\to\infty} E(t_n) = 0$. The same argument also applies to $f(v,\bar{t}_n)$ and gives $\lim_{n\to\infty} E(\bar{t}_n) = 0$.

Having proven (5.18) (under the condition (5.17)), we can now prove the timed averaging convergence (5.6) and the strong convergence (5.8) for $L^1$-norm $\|f(t)-M\|_{L^1}$. Suppose the assumptions in part (I) are satisfied. By the assumption (5.3) we have

$$\frac{1}{T}\int_0^T \sqrt{D(f(t))}\,dt \le \sqrt{\frac{1}{T}\int_0^T D(f(t))\,dt} \to 0 \quad (T\to\infty)\,.$$

Thus, from (5.16) we see that to prove (5.6) we need only to prove that

$$\frac{1}{T}\int_0^T E(t)dt \to 0 \quad (T\to\infty)\,. \tag{5.22}$$

We consider the following strategy: For any given $\varepsilon > 0$, choose a sequence of $T_n = T_{n,\varepsilon} \in [(2+t_0)^2,\infty)$ satisfying $T_n\to\infty$ $(n\to\infty)$ such that

$$\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^T \Big(E(t) - \varepsilon\|f(t)\|_{L^1_{2+}}\Big)\,dt = \lim_{n\to\infty}\frac{1}{|I_n|}\int_{I_n}\Big(E(t)-\varepsilon\|f(t)\|_{L^1_{2+}}\Big)\,dt \tag{5.23}$$

where $I_n = [\sqrt{T_n}\,,\,T_n]$ and we have used the boundedness

$$\sup_{t\ge0} E(t) \le 4|\mathbf{S}^{N-1}|\,,\quad C := \sup_{T\ge t_0}\frac{1}{T}\int_{t_0}^T \|f(t)\|_{L^1_{2+}}dt < \infty\,.$$

For every $n\in\mathbf{N}$ there exists $t_n\in I_n$ such that

$$\frac{1}{|I_n|}\int_{I_n}\Big(E(t)-\varepsilon\|f(t)\|_{L^1_{2+}}\Big)\,dt \le E(t_n) - \varepsilon\|f(t_n)\|_{L^1_{2+}}\,. \tag{5.24}$$

Since $T_n \ge (2+t_0)^2$, this gives an $L^1_{2+}$-bound (5.17):

$$\|f(t_n)\|_{L^1_{2+}} \le \frac{1}{\varepsilon}E(t_n) + \frac{2}{T_n}\int_{t_0}^{T_n}\|f(t)\|_{L^1_{2+}}dt \le \frac{4}{\varepsilon}|\mathbf{S}^{N-1}| + 2C\,.$$

Therefore there is a subsequence, still denote it as $\{t_n\}_{n\ge1}$, such that $E(t_n)\to 0$ as $n\to\infty$. By (5.23), (5.24) we thus obtain

$$\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^T E(t)dt \le \limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^T\Big(E(t)-\varepsilon\|f(t)\|_{L^1_{2+}}\Big)\,dt + \varepsilon C \le \varepsilon C\,.$$

Letting $\varepsilon\to0$ leads to (5.22).

Finally suppose the assumptions in part (II) are satisfied. Choose a sequence $\{t_n\}_{n\ge1}\subset[t_0,\infty)$ satisfying $t_n\to\infty$ such that

$$\limsup_{t\to\infty}\|f(t)-M\|_{L^1}=\lim_{n\to\infty}\|f(t_n)-M\|_{L^1}\,.$$

By the assumption in (5.7), $\sup_{n\ge1}\|f(t_n)\|_{L^1_{2+}}<\infty$. So there exist a subsequence, still denote it as $\{t_n\}_{n\ge1}$, and a sequence $\{\bar t_n\}_{n\ge1}$ satisfying $0\le\bar t_n-t_n\to0\,(n\to\infty)$, such that (5.18) holds true. Therefore by assumption (5.7) and applying (5.16) we have $\lim\limits_{n\to\infty}\|f(t_n)-f(\bar t_n)\|_{L^1}=0$ and

$$\begin{aligned}&\lim_{n\to\infty}\|f(t_n)-M\|_{L^1}\le\limsup_{n\to\infty}\|f(\bar t_n)-M\|_{L^1}\\&\le\frac{1}{L_R}\left(C_N\lim_{n\to\infty}\sqrt{D(f(\bar t_n))}+\lim_{n\to\infty}E(\bar t_n)\right)+\frac{2N}{R^2}=\frac{2N}{R^2}\qquad\forall\,0<R<\infty\,.\end{aligned}$$

This proves $\lim\limits_{n\to\infty}\|f(t_n)-M\|_{L^1}=0$ by letting $R\to\infty$. $\square$

**Proof of Theorem 1: Time Averaged Convergence**. This is a consequence of part (I) of Theorem 4 because the weak solution $f$ in Theorem 1 satisfies all conditions (5.1)-(5.5) with $\Phi(f)=|\log f|$ and $\Psi(v)=\langle v\rangle^{s+\gamma-2}$, where the uniform continuity (5.2) is indeed satisfied for any weak solution (see the proof of (3.5)). $\square$

## 6 Lower Bounds on the Convergence Rate

We first consider in the following two special cases (a) and (b) which motivate our work for general cases on the lower bounds of convergence rate.

**Case (a)**. $f|_{t=0}=f_0\in L^1_{(1,0,1)}(\mathbf{R}^N)$ and $f\in L^\infty([0,\infty);L^1_2(\mathbf{R}^N))$ is a mild solution of Eq.(B) with $B(z,\sigma)$ satisfying

$$\|B(z,\cdot)\|_{L^1(\mathbf{S}^{N-1})}\le K(1+|z|^2)^{\gamma/2}\,,\quad -\infty<\gamma<0\,.$$

**Case (b)**. $f|_{t=0}=f_0\in L^1_{(1,0,1)}\cap L^1\log L(\mathbf{R}^N)$ is an isotropic function and $f$ is an isotropic weak solution of Eq.(B) with $B(z,\sigma)$ satisfying

$$\|B(z,\cdot)\|_{L^1(\mathbf{S}^{N-1})}\le K|z|^\gamma\,,\quad -4\le\gamma\,,\ 1-N<\gamma<0\,.$$

Note that under the assumption in Case (a), the existence and uniqueness of mild solution $f$ is well-known and $f$ conserves the mass, momentum and energy. For Case (b), the existence of isotopic weak solution $f$ is also obvious.

**Theorem 5**. *For each Case (a) and Case (b) there are (explicit) constants* $0<C_1,C_2<\infty$ *depending only on* $N,K$ *and* $\gamma$ *such that*

$$\|f(t)-M\|_{L^1_2}\ge C_1\int_{|v|>t^\alpha}|v|^2f_0(v)dv-C_2\exp(-t^{2\alpha}/4)\qquad\forall\,t\ge0 \tag{6.1}$$

*where* $\alpha=1/\min\{|\gamma|\,,\,2\}$ *for Case (a) and* $\alpha=1/|\gamma|$ *for Case (b).*

*Proof.* Our proof relies on pointwise inequalities for mild solutions of Eq.(B). Recall that a non-negative measurable function $f(v,t)$ on $\mathbf{R}^N\times[0,\infty)$ is called a mild solution of Eq.(B) with initial datum $f_0$ if there is a null set $Z\subset\mathbf{R}^N$ which is independent of $t$ such that

$$\int_0^t Q^\pm(f)(v,\tau)d\tau<\infty\qquad\forall\,v\in\mathbf{R}^N\setminus Z\,,\quad\forall\,t\ge0$$

and

$$f(v,t) = f_0(v) + \int_0^t Q(f)(v,\tau)d\tau \qquad \forall\, v \in \mathbf{R}^N \setminus Z\,, \quad \forall\, t \geq 0\,.$$

Here $Q^+(f)$ and $Q^-(f) = fL(f)$ are defined in (5.10)-(5.13).

In the following, we use the letter $Z$ to denote a null set of $\mathbf{R}^N$ which is independent of $t$ – the particular null set may change from line to line. Recall also that if $f$ is a mild solution and satisfies that for almost every $v \in \mathbf{R}^N$ the function $t \mapsto L(f)(v,t)$ is locally integrable on $[0,\infty)$, then the the following integral formula holds:

$$f(v,t) = f_0(v)\exp(-\int_0^t L(f)(v,\tau)d\tau) + \int_0^t Q^+(f)(v,\tau)\exp(-\int_\tau^t L(f)(v,\tau_1)d\tau_1)d\tau$$

for all $t \geq 0$ and all $v \in \mathbf{R}^N \setminus Z$. This gives

$$f(v,t) \geq f_0(v)\exp\left(-\int_0^t L(f)(v,\tau)d\tau\right) \qquad \forall\, t \geq 0\,, \quad \forall\, v \in \mathbf{R}^N \setminus Z\,. \tag{6.2}$$

Case (a). By the assumption on $B$, we have

$$L(f)(v,t) \leq K \int_{\mathbf{R}^N} \frac{f(v_*,t)}{(1+|v-v_*|)^{|\gamma|}} dv_*\,.$$

Let $\beta = \min\{|\gamma|\,,\,2\}$. Then for all $v, v_* \in \mathbf{R}^N$,

$$\frac{1}{(1+|v-v_*|)^{|\gamma|}} \leq \frac{C_\beta}{|v|^\beta}\left(1+|v_*|^2\right)\ .$$

Hence, using conservation of mass and energy yields the bound

$$L(f)(v,t) \leq \frac{C_{K,\beta}\|f_0\|_{L^1_2}}{|v|^\beta} = \frac{c}{|v|^\beta}\,.$$

Thus,

$$f(v,t) \geq f_0(v)\exp(-\frac{c\,t}{|v|^\beta}) \qquad \forall\, t \geq 0\,, \quad \forall\, v \in \mathbf{R}^N \setminus Z\,,$$

and in particular,

$$f(v,t) \geq f_0(v)e^{-c} \qquad \forall\, t \geq 0\,, \quad \forall\, v \in \mathbf{R}^N \setminus Z \quad \text{s.t.} \quad |v| > t^\alpha \tag{6.3}$$

with $\alpha = 1/\beta$. Consequently,

$$\int_{|v|>t^\alpha} |v|^2 f(v,t)dv \geq e^{-c}\int_{|v|>t^\alpha} |v|^2 f_0(v)dv \qquad \forall\, t \geq 0\,.$$

Since $M(v) = (2\pi)^{-N/2}e^{-|v|^2/2}$, it follows that (6.1) holds true:

$$\begin{aligned}\|f(t) - M\|_{L^1_2} &\geq \int_{|v|>t^\alpha} |v|^2 f(v,t)dv - \int_{|v|>t^\alpha} |v|^2 M(v)dv \\ &\geq e^{-c}\int_{|v|>t^\alpha} |v|^2 f_0(v)dv - C_N\exp(-t^{2\alpha}/4) \qquad \forall\, t \geq 0\,.\end{aligned}$$

Case (b). We need to prove that under the assumptions in Case (b), $f$ is also a mild solution after a modification on a null set of $\mathbf{R}^N\times[0,+\infty)$. By $-4\le\gamma<0$ there is $m\in\{1,2\}$ such that $0\le m-|\gamma|/2\le 1$. We show that

$$\int_{\mathbf{R}^N}|v|^{2m}Q^{\pm}(f)(v,t)dv\le C\|f_0\|^2_{L^1_2}\qquad\forall\, t\ge 0\,. \tag{6.4}$$

where $C<\infty$ depends only on $K,\gamma$ and $N$. In fact using $\|B(v-v_*,\cdot)\|_{L^1(\mathbf{S}^{N-1})}\le K|v-v_*|^{\gamma}$ and $|v'|^2\le|v|^2+|v_*|^2$ we have

$$\int_{\mathbf{R}^N}|v|^{2m}Q^{\pm}(f)(v,t)dv\le K\int_{\mathbf{R}^N\times\mathbf{R}^N}\frac{f(v,t)f(v_*,t)(|v|^2+|v_*|^2)^m}{|v-v_*|^{|\gamma|}}dv_*dv\,.$$

Since $f$ is isotropic, i.e., $f(v,t)=f(|v|,t)$, and $|\gamma|<N-1$, it follows from Lemma 6.1 (see below) that

$$\int_{\mathbf{R}^N}\frac{f(v_*,t)(|v|^2+|v_*|^2)^m}{|v-v_*|^{|\gamma|}}dv_*\le C_{N,\gamma}\int_{\mathbf{R}^N}f(v_*,t)(|v|^2+|v_*|^2)^{m-|\gamma|/2}dv_*\,.$$

Since $(|v|^2+|v_*|^2)^{m-|\gamma|/2}\le\langle v\rangle^2\langle v_*\rangle^2$, (6.4) follows from conservation of mass and energy. By (6.4) we have

$$\int_0^T Q^{\pm}(f)(v,t)dt<\infty\qquad\forall\, v\in\mathbf{R}^N\setminus Z\,,\quad\forall\, T\in[0,\infty)\,.$$

Take any $\psi\in C_c^{\infty}(\mathbf{R}^N)$ and let

$$\varphi(v)=\rho(v)\psi(v)\,,\quad\rho(v)=\left(\frac{|v|^2}{1+|v|^2}\right)^m\,.$$

Then using (6.4) we have

$$\int_{\mathbf{R}^N}\varphi(v)Q^{\pm}(f)(v,t)dv\le C_{\psi}\|f_0\|^2_{L^1_2}<\infty\qquad\forall\, t\ge 0\,.$$

Therefore there is no integrability problem and we have with this $\varphi(v)=\psi(v)\rho(v)$

$$-\frac{1}{4}\int_0^t d\tau\int_{\mathbf{R}^N}Q(f\,|\Delta\varphi)(v,\tau)dv=\int_0^t d\tau\int_{\mathbf{R}^N}\varphi(v)Q(f)(v,t)dv\,.$$

Since $f$ is a weak solution, this gives for all $\psi\in C_c^{\infty}(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N}\psi(v)\rho(v)\left\{f(v,t)-f_0(v)-\int_0^t Q(f)(v,\tau)d\tau\right\}dv=0\quad\forall\, t\ge 0\,.$$

By $L^1$ -integrability of $\rho(v)\{\cdots\}$ and the strict positivity of $\rho(v)$ on $v\neq 0$, this implies

$$f(v,t)=f_0(v)+\int_0^t Q(f)(v,\tau)d\tau\qquad\forall\, t\ge 0\,,\quad\forall\, v\in\mathbf{R}^N\setminus Z_t$$

where $Z_t$ are null sets that may depend on $t$. Let

$$\bar{f}(v,t):=\left|f_0(v)+\int_0^t Q(f)(v,\tau)d\tau\right|\,,\quad(v,t)\in\mathbf{R}^N\times[0,\infty)\,.$$

Then it is not difficult to prove that $\bar{f}(v,t)$ is a mild solution of Eq.(B) and for any $t\in[0,\infty)$, $f(v,t)=\bar{f}(v,t)$ a.e. $v\in\mathbf{R}^N$. Thus we can replace $\bar{f}$ with $f$.

Now for all $v\in\mathbf{R}^N\setminus\{0\}$ we compute using Lemma 6.1 below and $\|f(t)\|_{L^1}=1$ that

$$L(f)(v,t)\le K\int_{\mathbf{R}^N}\frac{f(v_*,t)}{|v-v_*|^{|\gamma|}}dv_*\le\frac{c\|f(t)\|_{L^1}}{|v|^{|\gamma|}}=\frac{c}{|v|^{|\gamma|}}\qquad\forall\,t\ge 0\,.$$

Therefore (6.3) holds for $\alpha=1/|\gamma|$. This gives

$$\int_{|v|>t^\alpha}|v|^2f(v,t)dv\ge e^{-c}\int_{|v|>t^\alpha}|v|^2f_0(v)dv\qquad\forall\,t\ge 0$$

and thus, (as shown above) $f$ satisfies the inequality (6.1). □

**Lemma 6.1** *Let $f(v)=f(|v|)$ be a nonnegative isotropic measurable function on $\mathbf{R}^N$ with $N\ge 2$. Then For all $0\le\alpha,\beta<N-1$ and $v\in\mathbf{R}^N$*

$$\int_{\mathbf{R}^N}\frac{f(v_*)dv_*}{|v+v_*|^\alpha|v-v_*|^\beta}\le C_{N,\alpha,\beta}\int_{\mathbf{R}^N}\frac{f(v_*)dv_*}{(|v|^2+|v_*|^2)^{(\alpha+\beta)/2}}$$

*where* $C_{N,\alpha,\beta}=2^{(N+1)/2}\frac{|\mathbf{S}^{N-2}|}{|\mathbf{S}^{N-1}|}\left(\frac{1}{N-1-\alpha}+\frac{1}{N-1-\beta}\right)$.

*Proof.* For any $v\in\mathbf{R}^N$, let $v=\rho\omega,\rho\ge 0,\omega\in\mathbf{S}^{N-1}$. Then

$$\int_{\mathbf{R}^N}\frac{f(|v_*|)dv_*}{|v+v_*|^\alpha|v-v_*|^\beta}=\int_0^\infty r^{N-1}f(r)\left(\int_{\mathbf{S}^{N-1}}\frac{d\sigma}{|\rho\omega+r\sigma|^\alpha|\rho\omega-r\sigma|^\beta}\right)dr\,.$$

Since $\rho^2+r^2\pm 2\rho r\,t\ge\frac{1}{2}(\rho^2+r^2)(1-t^2)$ for $\pm t\le 0$, it follows that

$$\begin{aligned}&\int_{\mathbf{S}^{N-1}}\frac{d\sigma}{|\rho\omega+r\sigma|^\alpha|\rho\omega-r\sigma|^\beta}\\&\le\frac{|\mathbf{S}^{N-2}|}{(\rho^2+r^2)^{(\alpha+\beta)/2}}\left(2^{\alpha/2}\int_0^1(1-t^2)^{(N-3-\alpha)/2}dt+2^{\beta/2}\int_0^1(1-t^2)^{(N-3-\beta)/2}dt\right)\\&\le C_{N,\alpha,\beta}|\mathbf{S}^{N-1}|\frac{1}{(\rho^2+r^2)^{(\alpha+\beta)/2}}\,.\end{aligned}$$

This implies the inequality in the lemma. □

It is not clear whether the lower bound estimate (6.1) can be extended to all weak solutions. Our proof for Theorem 2 is very different from the above argument.

**Proof of Theorem 2**.

Part (I). By identity $|a-b|=b-a+2(a-b)^+$ ( with $(y)^+=\max\{y,0\}$) and conservation of mass and energy we have, for all $t\ge 0$, $R>0$,

$$\|f(t)-M\|_{L^1_2}\ge 2\int_{|v|>R}|v|^2f(v,t)dv-C_Ne^{-R^2/4}\tag{6.5}$$

where we used the exponential decay of $M(v)=(2\pi)^{-N/2}e^{-|v|^2/2}$. We now prove that there is finite constant $C_\gamma>0$ depending only on $N,\gamma,A^*,A_*,s,\|f_0\|_{L^1_s},H(f_0)$ and $K_0$ such that for all $R\ge 3$ and all $t\ge 0$

$$\int_{|v|>R}|v|^2f(v,t)dv\ge\int_{|v|>2R}|v|^2f_0(v)dv-\frac{C_\gamma(1+t)^{2-[2/s]}}{R^\beta}\,.\tag{6.6}$$

where $\beta = \min\{s,\ s-2+|\gamma|\}$. To prove (6.6), we use truncation. Let $\chi \in C^\infty(\mathbf{R}^N)$ be the function given in (3.1). For any $R > 1$, let $\psi_R(v) = |v|^2\chi(v/R)$. Then $\psi_R \in \mathcal{T}$,

$$|v|^2\mathbf{1}_{\{|v|>2R\}} \le \psi_R(v) \le |v|^2\mathbf{1}_{\{|v|>R\}} \qquad \forall\, v \in \mathbf{R}^N\,, \tag{6.7}$$

and $\sup\limits_{R>1}\|\partial^2\psi_R\|_{L^\infty} \le C_N$ . Also using $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ we have

$$\Delta\psi_R(v', v'_*, v, v_*) = \Delta\psi_R(v', v'_*, v, v_*)\mathbf{1}_{\{|v|^2+|v_*|^2>R^2\}}\,. \tag{6.8}$$

Suppose $-4 \le \gamma < -2$. Using the equation (1.11) to $\varphi = \psi_R$ we obtain from (6.7)

$$\begin{aligned}\int_{|v|>R}|v|^2 f(v,t)dv &\ge \int_{\mathbf{R}^N}\psi_R(v)f(v,t)dv\\ &\ge \int_{|v|>2R}|v|^2 f_0(v)dv - \frac{1}{4}\int_0^t d\tau\int_{\mathbf{R}^N}Q(f\,|\,\Delta\psi_R)(v,\tau)dv\,.\end{aligned}$$

Let $Q^1(\cdot,\cdot)$ and $L_1[\cdot]$ be the operators defined in Lemma 2.3 for $\lambda = 1$ corresponding to the kernels $B^1(z,\sigma) = \mathbf{1}_{\{|v-v_*|\le 1\}}B(z,\sigma)$ and $B_1(z,\sigma) = \mathbf{1}_{\{|z|>1\}}B(z,\sigma)$ respectively. Then by Lemma 2.3 we have

$$\begin{aligned}&\int_0^t d\tau\int_{\mathbf{R}^N}Q(f\,|\,\Delta\psi_R)(v,\tau)dv\\ &= \int_0^t d\tau\int_{\mathbf{R}^N}Q^1(f\,|\,\Delta\psi_R)(v,\tau)dv - 2\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}L_1[\Delta\psi_R]ff_*dv_*dv\,.\end{aligned}$$

We compute as in the proof of Lemma 2.2 that

$$\begin{aligned}&\left|\int_0^t d\tau\int_{\mathbf{R}^N}Q^1(f\,|\,\Delta\psi_R)(v,\tau)dv\right|\\ &\le C\int_0^t d\tau\left(\int_{|v|^2+|v_*|^2\ge R^2}|v-v_*|^{4+\gamma}\mathbf{1}_{\{|v-v_*|\le 1\}}ff_*dv_*dv\right)^{1/2}\sqrt{D(f(\tau))}\\ &\le C\left(\int_0^t d\tau\int_{|v|^2+|v_*|^2\ge R^2}\mathbf{1}_{\{|v-v_*|\le 1\}}ff_*dv_*dv\right)^{1/2}\end{aligned}$$

By $R \ge 3$, we see that $|v| > R/\sqrt{2}$ and $|v - v_*| \le 1$ imply $|v_*| > R/3$ and so

$$\begin{aligned}&\int_0^t d\tau\int_{|v|^2+|v_*|^2\ge R^2}\mathbf{1}_{\{|v-v_*|\le 1\}}ff_*dv_*dv \le 2\int_0^t d\tau\left(\int_{|v|\ge R/3}f(v,\tau)dv\right)^2\\ &\le \frac{C}{R^{2s}}\int_0^t(1+\tau)^{2-2[2/s]}d\tau \le \frac{C(1+t)^{3-2[2/s]}}{R^{2s}}\end{aligned}$$

where we have used the moment estimates (for $s > 2$) and the conservation of mass and energy (for $s = 2$). Thus

$$\left|\int_0^t d\tau\int_{\mathbf{R}^N}Q^1(f\,|\,\Delta\psi_R)(v,\tau)dv\right| \le \frac{C(1+t)^{3/2-[2/s]}}{R^s}\,.$$

Also by $2+\gamma<0$ we have

$$\left|\int_0^t d\tau\int_{\mathbf{R}^N} L_1[\Delta\psi_R]ff_*dv\right|\le C\int_0^t d\tau\int_{|v|^2+|v_*|^2\ge R^2}|v-v_*|^{2+\gamma}\mathbf{1}_{\{|v-v_*|>1\}}ff_*dv_*dv$$
$$\le C\int_0^t d\tau\int_{|v|\ge R/\sqrt{2}}f(v,\tau)dv\le\frac{C}{R^s}\int_0^t(1+\tau)^{1-[2/s]}d\tau\le\frac{C(1+t)^{2-[2/s]}}{R^s}\,.$$

Therefore

$$\int_{|v|>R}|v|^2f(v,t)dv\ge\int_{|v|>2R}|v|^2f_0(v)dv-\frac{C_\gamma(1+t)^{2-[2/s]}}{R^s}\,. \tag{6.9}$$

Next suppose $-2\le\gamma<0$. In this case we use (6.8) and moment estimates to get

$$\left|\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}L[\Delta\psi_R]ff_*dv_*v\right|\le C\int_0^t d\tau\int_{|v|^2+|v_*|^2>R^2}|v-v_*|^{2+\gamma}ff_*dv_*dv$$
$$\le\frac{C}{R^{s-2+|\gamma|}}\int_0^t d\tau\int_{\mathbf{R}^N\times\mathbf{R}^N}(|v|^2+|v_*|^2)^{s/2}ff_*dv_*dv\le C\frac{(1+t)^{2-[2/s]}}{R^{s-2+|\gamma|}}\,,\quad t\ge0\,.$$

Therefore using equation (1.12) with $\varphi=\psi_R$ we obtain

$$\int_{|v|>R}|v|^2f(v,t)dv\ge\int_{|v|>2R}|v|^2f_0(v)dv-\frac{C_\gamma(1+t)^{2-[2/s]}}{R^{s-2+|\gamma|}}\,. \tag{6.10}$$

The inequality (6.6) follows from (6.9) and (6.10).

Combining (6.5) and (6.6) and using $e^{-R^2/4}\le\frac{C}{R^\beta}\le\frac{C(1+t)^{2-[2/s]}}{R^\beta}$ we get with a larger constant $0<K<\infty$ such that for all $R\ge3$

$$\|f(t)-M\|_{L^1_2}\ge2\int_{|v|>2R}|v|^2f_0(v)dv-\frac{K(1+t)^{2-[2/s]}}{(2R)^\beta}\qquad t\ge0\,. \tag{6.11}$$

We can assume $K\ge\max\{K_0,\,6^\beta N\}$ so that the solution $\mathrm{R}(t)$ of the equation (1.19) satisfies $\mathrm{R}(t)\ge6$ for all $t\ge0$. Therefore inserting $R=\frac{1}{2}\mathrm{R}(t)$ into (6.11) and using equation (1.19) gives (1.20).

Part (II). Recall the assumptions in the theorem. We have for all $R\ge R_0$

$$\int_{|v|>R}f_0(v)|v|^2dv\ge C\int_R^\infty r^{-1-\delta}dr=CR^{-\delta}. \tag{6.12}$$

Since $\delta<\beta$, $f_0$ satisfies (1.18). By Part(I) of Theorem 2, the solution $f(v,t)$ satisfies (1.20) with the function $\mathrm{R}(t)$ defined through (1.19) for some constant $K\ge N(R_0)^\beta$ which implies that $\mathrm{R}(t)\ge R_0$ for all $t\ge0$. By (1.19) and (6.12) with $R=\mathrm{R}(t)$ we have

$$K(1+t)^2\ge C(\mathrm{R}(t))^{\beta-\delta}\,.$$

So $\mathrm{R}(t)\le C(1+t)^{2/(\beta-\delta)}$ and hence using (6.12) with $R=\mathrm{R}(t)$ again we obtain

$$\int_{|v|>\mathrm{R}(t)}f_0(v)|v|^2dv\ge C(\mathrm{R}(t))^{-\delta}\ge C(1+t)^{-2\delta/(\beta-\delta)}\,,\quad t\ge0\,.$$

This proves (1.22).

Part (III). Let $s = 2$. By the assumption on $f_0$, and recalling that $A_1(t) = -\frac{d}{dt}A(t) \geq 0$, we have

$$\int_{|v|>R} f_0(v)|v|^2 dv \geq C \int_R^\infty A_1(r)dr = CA(R) \qquad \forall\, R \geq R_0. \tag{6.13}$$

On the other hand, by assumptions on $A(t)$, there is a constant $C > 0$ such that $A(R) \geq CR^{-\delta}$ for all $R \geq R_0$. Since $\delta < \beta$, this implies that $f_0$ satisfies (1.18) and thus from the equation (1.19) (with some constant $K \geq N(R_0)^\beta$) we get as shown above that $\mathrm{R}(t) \geq R_0$ and

$$K(1+t) \geq C(\mathrm{R}(t))^{\beta-\delta}\,, \qquad t \geq 0\,.$$

So $\mathrm{R}(t) \leq c(1+t)^\alpha$ with $\alpha = 1/(\beta - \delta)$. Applying (6.13) with $R = \mathrm{R}(t)$, and recalling that $A(t)$ is non-increasing, we obtain

$$\int_{|v|>\mathrm{R}(t)} f_0(v)|v|^2 dv \geq CA(c(1+t)^\alpha)\,, \quad t \geq 0\,.$$

Finally by assumption (1.23) on $A(t)$ it is easily proved that there are constants $0 < c_1, C_1 < \infty$ such that $A(c(1+t)^\alpha) \geq C_1 A(c_1 t^\alpha)$ for all $t \geq 0$. This proves (1.25). $\square$

# 7 Upper Bounds on the Convergence Rate for $\gamma \geq -1$ and Grad Angular Cutoff

This section is devoted to the proof of Theorem 3. There are several somewhat long and technical arguments, as well as some that are shorter and more plainly motivated. We begin with one of the more technical lemmas which shall later in this section be used to bound the *entropic moments* as explained in the introduction. It is through this lemma that the condition $\gamma \geq -1$ enters Theorem 3. Next, in Lemma 7.2, we prove the Gronwall type lemma that we shall use to get power law convergence out of the entropy production estimate that we shall eventually prove here, and then, with the crucial preliminaries behinds us, in Lemma 7.3, we recall Villani's entropy production bound for super hard potentials, and explain how we shall use it here. On a first reading of this section, the reader may wish to turn to Lemma 7.3, and start from there, though in our exposition we now turn to Lemma 7.1.

Instead of considering directly the growth of entropic moments defined in terms of $\langle v\rangle^k$, we shall instead use a more symmetric function $S$:

$$S(v, v_*, v', v'_*) = \min\{\max\{\Phi(v)\,, \Phi(v_*)\}\,,\, \max\{\Phi(v')\,, \Phi(v'_*)\}\} \tag{7.1}$$

where

$$\Phi(v) = \min\{\langle v\rangle^k\,, R\}\,, \quad k > 0\,, R > 0\,. \tag{7.2}$$

The idea is that the difference $\Phi - S$ contributes a factor $|v - v_*|$ that kills the singularity $|v - v_*|^\gamma$ of $B(v - v_*, \sigma)$ and this is why we assume that $-1 \leq \gamma < 0$, while the integral involved $S$ can be treated as an entropy dissipation. Also to overcome the problem of $f_0$ having no pointwise lower bounds we consider a suitable convex combination of the solution and the Maxwellian.

We need several lemmas.

**Lemma 7.1.** *Let $b(\cdot) \ge 0$ satisfy (1.27), $0 < \beta \le 1$, and let $\Phi$, $S$ be introduced above with $k \ge 1$. Then for all $0 \le f \in L^1_k \cap L^1\log L(\mathbf{R}^N)$ we have (with $\cos\theta = \langle (v-v_*)/|v-v_*|, \sigma\rangle$)*

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} \frac{b(\cos\theta)}{|v-v_*|^\beta}\Big(S(v,v_*,v',v_*') - \Phi(v)\Big)^+ f f_* \log^+ f \, d\sigma dv_* dv$$
$$\le 2kA_0\left(\|f\log^+ f\|_{L^1} + \|f\|_{L^1}\right)\|f\|_{L^1_{k-\beta}}\,. \tag{7.3}$$

$$\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} \frac{b(\cos\theta)}{|v-v_*|^\beta}\Big(\Phi(v') - S(v,v_*,v',v_*')\Big)^+ f f_* \log^+ f' \, d\sigma dv_* dv$$
$$\le 4^k N A_0\left(\|f\log^+ f\|_{L^1} + \|f\|_{L^1}\right)\|f\|_{L^1_{k-\beta}}\,. \tag{7.4}$$

*where $(y)^+ = \max\{y, 0\}$ and $A_0$ is given in (1.27).*

Note that the right hand side of the inequalities do not depend on $R$, so letting $R\to\infty$ one sees that (7.3)-(7.4) hold also for $\Phi(v) = \langle v\rangle^k$ by Fatou's Lemma.

To prove the lemma we will use the following formula which are results of changing variables: Let $W(t) \ge 0, f(v) \ge 0$ be measurable on $[-1,1]$ and $\mathbf{R}^N$ respectively. Then for all $v \in \mathbf{R}^N$

$$\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} W(\cos\theta) f(v') d\sigma dv_* = \left(|\mathbf{S}^{N-2}|\int_0^\pi \frac{W(\cos\theta)\sin^{N-2}\theta}{\sin^N(\theta/2)}d\theta\right)\int_{\mathbf{R}^N} f(v_*)dv_*\,, \tag{7.5}$$

$$\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} W(\cos\theta) f(v_*') d\sigma dv_* = \left(|\mathbf{S}^{N-2}|\int_0^\pi \frac{W(\cos\theta)\sin^{N-2}\theta}{\cos^N(\theta/2)}d\theta\right)\int_{\mathbf{R}^N} f(v_*)dv_*\,. \tag{7.6}$$

**Proof of Lemma 7.1.** We first prove the following inequalities:

$$\Big(S(v,v_*,v',v_*') - \Phi(v)\Big)^+ \le 2k\langle v_*\rangle^{k-\beta}|v-v_*|^\beta\,, \tag{7.7}$$

$$\Big(\Phi(v') - S(v,v_*,v',v_*')\Big)^+ \le k2^k\Big(\langle v\rangle^{k-\beta} + \langle v_*\rangle^{k-\beta}\Big)|v-v_*|^\beta m(\theta)\,, \tag{7.8}$$

$$m(\theta) f_*\left(\log^+ f' + \log^+ f_*'\right) \le 2f_*\log^+ f_* + 2Nf_* + (m(\theta))^N\left(f'\log^+ f' + f_*'\log^+ f_*'\right)\,, \tag{7.9}$$

where

$$m(\theta) = \min\{\sin(\theta/2)\,,\, \cos(\theta/2)\}\,.$$

Given any $(v,v_*,\sigma) \in \mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}$. Suppose $\Phi(v) < S(v,v_*,v',v_*')$. In this case we have by definition of $S$ that $S(v,v_*,v',v_*') \le \max\{\Phi(v),\,\Phi(v_*)\}$. Consequently, $\langle v\rangle < \langle v_*\rangle$ and so, $|v-v_*| \le |v| + |v_*| \le 2\langle v_*\rangle$. Therefore,

$$0 < S(v,v_*,v',v_*') - \Phi(v) \le k\langle v_*\rangle^{k-1}|v-v_*| \le k2\langle v_*\rangle^{k-\beta}|v-v_*|^\beta\,.$$

The other case to consider is slightly more involved. Suppose $\Phi(v') > S(v,v_*,v',v_*')$. Then,

$$S(v,v_*,v',v_*') = \max\{\Phi(v),\,\Phi(v_*)\}\,.$$

Since $|v'-v| = |v-v_*|\sin(\theta/2)$, $|v'-v_*| = |v-v_*|\cos(\theta/2)$, it follows that

$$0 < \Phi(v') - S(v,v_*,v',v_*') = \min\{\Phi(v') - \Phi(v)\,,\, \Phi(v') - \Phi(v_*)\}$$
$$\le k2^k\left(\langle v\rangle^{k-\beta} + \langle v_*\rangle^{k-\beta}\right)|v-v_*|^\beta m(\theta)\,.$$

Next, if $f(v') \le f(v_*)(m(\theta))^{-N}$, then

$$\log^+ f(v') \le \log^+ f(v_*) + N|\log m(\theta)|$$

so that (using $x|\log x| \le 1$ for $0 \le x \le 1$)

$$\begin{aligned}&m(\theta)f(v_*)\log^+ f(v')\\ &\le m(\theta)f(v_*)\log^+ f(v_*) + Nm(\theta)|\log m(\theta)|f(v_*)\\ &\le f(v_*)\log^+ f(v_*) + Nf(v_*)\,.\end{aligned}$$

If $f(v') \ge f(v_*)(m(\theta))^{-N}$, then $f(v_*) \le (m(\theta))^N f(v')$ so that

$$m(\theta)f(v_*)\log^+ f(v') \le (m(\theta))^N f(v')\log^+ f(v')\,.$$

Thus

$$m(\theta)f_*\log^+ f' \le f_*\log^+ f_* + Nf_* + (m(\theta))^N f'\log^+ f'\,.$$

With the same argument one has

$$m(\theta)f_*\log^+ f'_* \le f_*\log^+ f_* + Nf_* + (m(\theta))^N f'_*\log^+ f'_*\,.$$

So the inequalities (7.7)-(7.9) hold.

We now prove (7.3). From (7.7)-(7.9) we get

$$\begin{aligned}&\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} \frac{b(\cos\theta)}{|v-v_*|^\beta}\Big(S(v,v_*,v',v'_*)-\Phi(v)\Big)^+ ff_*\log^+ f\,d\sigma dv_* dv\\ &\le 2kA_0\int_{\mathbf{R}^N\times\mathbf{R}^N}\langle v_*\rangle^{k-\beta} ff_*\log^+ f\,dv_*dv = 2kA_0\|f\log^+ f\|_{L^1}\|f\|_{L^1_{k-\beta}}\,,\end{aligned}$$

Next, let $I(f)$ denote the integral on the left side of (7.4). Then

$$\begin{aligned}I(f) &:= \int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} \frac{b(\cos\theta)}{|v-v_*|^\beta}\Big(\Phi(v')-S(v,v_*,v',v'_*)\Big)^+ ff_*\log^+ f'\,d\sigma dv_* dv\\ &\le k2^k\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} b(\cos\theta)m(\theta)\left(\langle v\rangle^{k-\beta}+\langle v_*\rangle^{k-\beta}\right) ff_*\log^+ f'\,d\sigma dv_* dv\\ &= k2^k\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} b(\cos\theta)m(\theta)f_*(\log^+ f'+\log^+ f'_*)d\sigma dv_*\right)\langle v\rangle^{k-\beta} f\,dv\\ &\le k2^k\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} b(\cos\theta)(2f_*\log^+ f_* + 2Nf_*)d\sigma dv_*\right)\langle v\rangle^{k-\beta} f\,dv\\ &+k2^k\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} b(\cos\theta)(m(\theta))^N(f'\log^+ f' + f'_*\log^+ f'_*)d\sigma dv_*\right)\langle v\rangle^{k-\beta} f\,dv\\ &:= I_1 + I_2\,.\end{aligned}$$

Evidently,

$$I_1 \le k2^k A_0\left(2\|f\log^+ f\|_{L^1} + 2N\|f\|_{L^1}\right)\|f\|_{L^1_{k-\beta}}\,.$$

To estimate $I_2$, we use the formulas (7.5)-(7.6) to compute the inner integral

$$\begin{aligned}
&\int_{\mathbf{R}^N\times\mathbf{S}^{N-1}} b(\cos\theta)(m(\theta))^N(f'\log^+ f' + f'_*\log^+ f'_*)d\sigma dv_*\\
&= |\mathbf{S}^{N-2}|\int_0^\pi b(\cos\theta)\sin^{N-2}\theta\left(\frac{(m(\theta))^N}{\sin^N(\theta/2)}+\frac{(m(\theta))^N}{\cos^N(\theta/2)}\right)d\theta\|f\log^+ f\|_{L^1}\\
&\le 2|\mathbf{S}^{N-2}|\int_0^\pi b(\cos\theta)\sin^{N-1}\theta d\theta\|f\log^+ f\|_{L^1} = 2A_0\|f\log^+ f\|_{L^1}\,.
\end{aligned}$$

Consequently,

$$I_2 \le 2A_0k2^k\|f\log^+ f\|_{L^1}\int_{\mathbf{R}^N}\langle v\rangle^{k-\beta}f\,dv = 2A_0k2^k\|f\log^+ f\|_{L^1}\|f\|_{L^1_{k-\beta}}\,.$$

Combining the above estimates,

$$\begin{aligned}
I(f) &\le k2^kA_0\left(4\|f\log^+ f\|_{L^1}+2N\|f\|_{L^1}\right)\|f\|_{L^1_{k-\beta}}\\
&\le 2^{2k}NA_0\left(\|f\log^+ f\|_{L^1}+\|f\|_{L^1}\right)\|f\|_{L^1_{k-\beta}}\,.
\end{aligned}$$

□

**Lemma 7.2.** *Let $u(t)\ge 0$ defined on $[0,\infty)$ be absolutely continuous on $[0,T]$ for all $0<T<\infty$ and satisfy for some constants $C_1>0$, $C_2\ge 0$, $k\ge 0\,,\varepsilon>0$, $\eta<1$,*

$$\frac{d}{dt}u(t)\le -C_1(1+t)^{-\eta}[u(t)]^{1+\varepsilon}+C_2(1+t)^ke^{-t}\,,\qquad \text{a.e.}\quad t\ge 0\,.$$

*Then there is a constant $0<C<\infty$ depending only on $C_1$, $C_2$, $k\,,\varepsilon,\eta$, and $u(0)$, such that*

$$u(t)\le C(1+t)^{-\alpha}\qquad \forall\, t\ge 0$$

*where $\alpha=\dfrac{1-\eta}{\varepsilon}$.*

**Proof**. Choose a constant $C\ge\max\{u(0),1\}$ large enough such that

$$C^\varepsilon\ge\frac{1}{C_1}(\alpha+C_2(1+t)^{k+\alpha+1}e^{-t})\qquad\forall\, t\ge 0\,.$$

Let

$$U(t)=C(1+t)^{-\alpha}\,,\quad t\ge 0\,.$$

Then using $\alpha+1=\alpha(\varepsilon+1)+\eta$ and $C\ge 1$ we compute

$$\begin{aligned}
&\frac{dU(t)}{dt}+C_1(1+t)^{-\eta}[U(t)]^{1+\varepsilon}-C_2(1+t)^ke^{-t}\\
&=(1+t)^{-\alpha-1}\left(C_1C^{1+\varepsilon}-C\alpha\right)-C_2(1+t)^ke^{-t}\\
&\ge(1+t)^{-\alpha-1}C\Big(C_1C^\varepsilon-\alpha-C_2(1+t)^{k+\alpha+1}e^{-t}\Big)\ge 0\,.
\end{aligned}$$

By the absolute continuity of $u(t)$, and $u(0)\le U(0)$, we have for any $t>0$,

$$\begin{aligned}
&\Big(u(t)-U(t)\Big)^+=\int_0^t\left(\frac{d}{d\tau}u(\tau)-\frac{d}{d\tau}U(\tau)\right)\mathbf{1}_{\{u(\tau)>U(\tau)\}}d\tau\\
&\le\int_0^t\Big(-C_1(1+\tau)^{-\eta}[u(\tau)]^{1+\varepsilon}+C_2(1+\tau)^ke^{-\tau}-\frac{d}{d\tau}U(\tau)\Big)\mathbf{1}_{\{u(\tau)>U(\tau)\}}d\tau\\
&\le\int_0^t\Big(-C_1(1+\tau)^{-\eta}[U(\tau)]^{1+\varepsilon}+C_2(1+\tau)^ke^{-\tau}-\frac{d}{d\tau}U(\tau)\Big)\mathbf{1}_{\{u(\tau)>U(\tau)\}}d\tau\le 0\,.
\end{aligned}$$

So $u(t) \leq U(t) \ \ \forall\, t \geq 0\,.$ $\square$

**Lemma 7.3** (Villani's Inequality [24]). *Let*

$$\mathcal{D}_2(f) = \frac{1}{4}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} (1+|v-v_*|^2)(ff_* - f'f'_*)\log\left(\frac{ff_*}{f'f'_*}\right) d\sigma dv dv_*$$

*and let $M \in L^1_{(1,0,1)}(\mathbf{R}^N)$ be the Maxwellian (1.9). Then for all $f \in L^1_{(1,0,1)} \cap L^1\log L(\mathbf{R}^N)$,*

$$\frac{|\mathbf{S}^{N-1}|}{4(2N+1)}(N - T_f^*)H(f|M) \leq \mathcal{D}_2(f)$$

*where $H(f|M)$ is the relative entropy:*

$$H(f|M) = \int_{\mathbf{R}^N} f\log(f/M)dv = H(f) - H(M) \geq 0$$

*and*

$$T_f^* = \max_{\mathbf{e}\in\mathbf{S}^{N-1}} \int_{\mathbf{R}^N} \langle v, \mathbf{e}\rangle^2 f(v)dv\,.$$

*Moreover for any $H_0 \in [0, +\infty)$*

$$\inf_{f\in\mathcal{H}_0}(N - T_f^*) > 0 \tag{7.10}$$

*where $\mathcal{H}_0 = \{f \in L^1_{(1,0,1)} \cap L^1\log L(\mathbf{R}^N)\,|\, H(f|M) \leq H_0\}$.*

This result is Theorem 2.1 in [24], except for the lower bound (7.10), which is an elaboration of the remark following Theorem 2.1 in [24]. There it is observed that for $T_f^* = N$ to hold, $f$ would have to be concentrated on a line, which is inconsistent with the finite entropy hypothesis. That a uniform bound of the type (7.10) holds was communicated to us by Villani, along with a sketched proof, and so we list it here as part of his theorem, and provide a detailed proof of his bound:

First of all, one has the relation

$$\inf_{f\in\mathcal{H}_0}(N - T_f^*) = \inf_{f\in\mathcal{H}_0}\int_{\mathbf{R}^N}|\hat{v}|^2 f(v)dv\,, \quad \hat{v} = (v_2, ..., v_N)\,.$$

Take $\varepsilon > 0, R > 0, K > 1$ and consider

$$\int_{\mathbf{R}^N}|\hat{v}|^2 f(v)dv \geq \varepsilon^2\int_{|\hat{v}|\geq\varepsilon} f(v)dv = \varepsilon^2\left(1 - \int_{|\hat{v}|<\varepsilon} f(v)dv\right)\,,$$

and

$$\begin{aligned}\int_{|\hat{v}|<\varepsilon} f(v)dv &\leq \int_{|\hat{v}|<\varepsilon, |v_1|\leq R, f(v)\leq K} f(v)dv + \int_{|v_1|>R} f(v)dv + \int_{f(v)>K} f(v)dv \\ &\leq K2R(2\varepsilon)^{N-1} + \frac{N}{R^2} + \int_{f(v)>K} f(v)dv\,.\end{aligned}$$

To estimate the remaining integral, we use Young's inequality in the form

$$ab \leq ca\log a + ce^{b/c-1}$$

valid for all $a \geq 0, b \geq 0, c > 0$. (The $c = 1$ case is standard. To generalize this to the present case, replace $b$ by $b/c$, and multiply through by $c$.) Taking $a = f/M$ and $b = 1_{\{f(v)>K\}}$, and keeping $c$ arbitrary for the moment, integrating this inequallity against $M$ this leads to

$$\begin{aligned}
\int_{f(v)>K} f(v)dv &\leq c\int_{\mathbf{R}^N}\left(\frac{f}{M}\right)\log\left(\frac{f}{M}\right)M(v)dvdv + c\int_{\mathbf{R}^3} e^{(1/c)1_{\{f(v)>K\}}-1}M(v)dv \\
&= cH(f|M) + \frac{c}{e}\int_{f(v)\leq K} M(v)dv + c\int_{f(v)>K} e^{(1/c)-1}M(v)dv \\
&\leq cH_0 + \frac{c}{e} + \frac{c}{K}e^{(1/c)-1} := C_{c,K,H_0}
\end{aligned} \tag{7.11}$$

where in the last line, we have used the $H$-theorem, the fact that $M < 1$, and Chebychev's inequality to estimate the Lebesgue measure of $\{f(v) > K\}$.

Thus

$$\int_{|\hat{v}|<\varepsilon} f(v)dv \leq 2^N KR\varepsilon^{N-1} + \frac{N}{R^2} + C_{c,K,H_0}\,. \tag{7.12}$$

With suitable choice of $c > 0$, $K > 1, R > 1$ and $\varepsilon > 0$, the right hand side of (7.12) is less than 1/2. This gives (7.10).

**Proof of Theorem 3.** First of all, using inequality (1.15) and the Csiszar-Kullback inequality $\|f - M\|_{L^1} \leq \sqrt{2H(f|M)}$, we have

$$\|f - M\|_{L^1_2} \leq C_N[H(f|M)]^{1/4} \qquad \forall\, f \in L^1_{(1,0,1)} \cap L^1\log L(\mathbf{R}^N)\,. \tag{7.13}$$

¿From (7.13), we see that to prove the theorem, it suffices to prove that there is a weak solution $f$ of Eq.(B) with $f|_{t=0} = f_0$ such that for some constant $0 < C < \infty$

$$H(f(t)|M) \leq C(1+t)^{-4\lambda} \qquad \forall\, t \geq 0 \tag{7.14}$$

where $\lambda$ is given in (1.29).

The proof consists of three steps.

**Step 1**. In the first two steps we assume in addition that $B(z,\sigma)$ is bounded: $B(z,\sigma) \leq$ const. In this case it is well-known that Eq.(B) has a unique mild solution $f \in C^1([0,\infty); L^1(\mathbf{R}^N)) \cap L^\infty([0,\infty); L^1_2 \cap L^1\log L(\mathbf{R}^N))$ satisfying $f|_{t=0} = f_0$, and moreover, $f$ conserves the mass, moment and energy, and satisfies the entropy identity (see e.g. [17] ). Hence by Proposition 1.1 (b), the boundedness of $B(z,\sigma)$ implies that the mild solution $f$ is also a weak solution, and thus by Theorem 1, it satisfies the moment estimate:

$$\|f(t)\|_{L^1_s} \leq C(1+t)\,, \qquad \int_0^t \|f(\tau)\|_{L^1_{s-|\gamma|}} d\tau \leq C(1+t)\,, \quad t \geq 0\,.$$

where the constant $C$ is given in Theorem 1, in particular it does not depend on the $L^\infty$ bound of $B(z,\sigma)$.

To overcome the trouble that $f$ may not have a lower bound, we consider a suitable convex combination of $f$ and the Maxwellian $M$:

$$g(v,t) = (1 - e^{-t-1})f(v,t) + e^{-t-1}M(v)\,. \tag{7.15}$$

It is obvious that the flow $t \mapsto g(t)$ has the same mass, momentum, and energy as $f(t)$ and holds the following properties that will be proven in this step:

$$\log^+ g \le \log^+ f\,, \quad g\log^+ g \le f\log^+ f\,, \tag{7.16}$$

$$\log^+(1/g(v,t)) \le C(1+t)\langle v\rangle^2\,, \tag{7.17}$$

$$H(f(t)|M) \le H(g(t)|M) + C(1+t)e^{-t}\,, \tag{7.18}$$

$$\frac{d}{dt}H(g(t)|M) \le -D(g(t)) + C(1+t)e^{-t} \quad \text{a.e.} \quad t\ge 0\,, \tag{7.19}$$

$$\|g(t)\log^+ g(t)\|_{L^1_k} \le C(1+t)^2\,. \qquad (\, k = s-2\,) \tag{7.20}$$

Here and below all constants $0 < C < \infty$ depend only $N, K_*, A_0, \gamma, , s, \|f_0\|_{L^1_s}$ and $\|f_0\log f_0\|_{L^1_s}$.

Proof of (7.16)-(7.18): By $M(v) = (2\pi)^{-N/2}e^{-|v|^2/2} < 1$ we have

$$g(v,t)\ge 1 \implies g(v,t)\le f(v,t) \implies 0\le \log g(v,t) \le \log f(v,t)\,.$$

So (7.16) is true. (7.17) is obvious. To prove (7.18) we denote $\delta = e^{-t-1}$. Then

$$\begin{aligned} H(g(t)) &\ge \int_{\mathbf{R}^N}(1-\delta)f\log((1-\delta)f)dv + \int_{\mathbf{R}^N}\delta M\log(\delta M)dv \\ &= (1-\delta)\log(1-\delta) + (1-\delta)H(f(t)) + \delta\log\delta + \delta H(M)\,. \end{aligned}$$

So using $(1-\delta)\log(\frac{1}{1-\delta}) \le \delta$ and $0\le H(f(t)|M) \le H(f_0|M)$ we obtain (7.18):

$$\begin{aligned} H(f(t)|M) &\le H(g(t)|M) + (1-\delta)\log(\frac{1}{1-\delta}) + \delta H(f(t)|M) + \delta\log(1/\delta) \\ &\le H(g(t)|M) + [H(f_0|M) + 2 + t]e^{-t-1}\,. \end{aligned}$$

Now we are going to prove (7.19). Since $f(v,t)$ is a mild solution and $g(v,t)\ge e^{-t-1}M(v)$, the function $t\mapsto g(v,t)\log g(v,t)$ is also absolutely continuous on any finite interval for almost every $v\in\mathbf{R}^N$. So we have for almost every $v\in\mathbf{R}^N$ and for all $t\ge 0$

$$\begin{aligned} g(v,t)\log g(v,t) &= g_0(v)\log g_0(v) \\ &+ \int_0^t (1+\log g(v,\tau))\Big(e^{-\tau-1}(f(v,\tau)-M(v)) + (1-e^{-\tau-1})Q(f)(v,\tau)\Big)d\tau\,. \end{aligned} \tag{7.21}$$

We need to show that there are no problems of integrability in $v\in\mathbf{R}^N$. In fact, it is easily seen that the functions

$$\begin{aligned} & g(v,t)|\log g(v,t)|\,, \quad |g_0(v)|\log g_0(v)|\,, \\ & \int_0^t (1+|\log g(v,\tau)|)(f(v,\tau)+M(v))d\tau\,, \\ & \int_0^t (1+\log^+(1/g(v,\tau))[Q^+(f)(v,\tau) + Q^-(f)(v,\tau)]d\tau\,, \\ & \int_0^t Q^-(f)(v,\tau)\log^+ g(v,\tau)d\tau \end{aligned}$$

are all integrable on $\mathbf{R}^N$, while from the equation (7.21) we see that the rest term

$$\int_0^t (1-e^{-\tau-1})Q^+(f)(v,\tau)\log^+ g(v,\tau)d\tau$$

is bounded by the summation of the above functions and thus it is also integrable on $\mathbf{R}^N$. Since $1-e^{-\tau-1} > 1/2$, it follows that

$$\int_0^t\int_{\mathbf{R}^N} Q^+(f)(v,\tau)\log^+ g(v,\tau)dvd\tau < \infty\,.$$

Therefore there are no problems of integrability and we obtain for any $0\le\varphi\in L^\infty(\mathbf{R}^N)$ and for all $t\ge 0$

$$\begin{aligned}&\int_{\mathbf{R}^N}\varphi(v)g(v,t)\log g(v,t)dv = \int_{\mathbf{R}^N}\varphi(v)g_0(v)\log g_0(v)dv\\ &+\int_0^t d\tau\int_{\mathbf{R}^N}\varphi(v)\Big(e^{-\tau-1}(f(v,\tau)-M(v))+(1-e^{-\tau-1})Q(f)(v,\tau)\Big)(1+\log g(v,\tau))dv\,.\end{aligned} \tag{7.22}$$

In particular taking $\varphi(v)\equiv 1$ we see that $t\mapsto H(g(t))$ is absolutely continuous on every finite intervals and using conservation of mass and energy we have

$$\begin{aligned}\frac{d}{dt}H(g(t)|M) &= e^{-t-1}\int_{\mathbf{R}^N}(f(v,t)-M(v))\log g(v,t)dv\\ &+ (1-e^{-t-1})\int_{\mathbf{R}^N}Q(f)(v,t)\log g(v,t))\,dv \quad \text{a.e.}\quad t\ge 0\,.\end{aligned} \tag{7.23}$$

Since $\sup_{t\ge 0}\|f(t)\log f(t)\|_{L^1}\le H(f_0)+C_N$, it follows that

$$\begin{aligned}&\int_{\mathbf{R}^N}(f(v,t)-M(v))\log g(v,t))dv = \int_{\mathbf{R}^N} f\log g dv + \int_{\mathbf{R}^N} M\log(1/g)dv\\ &\le \|f(t)\log^+ f(t)\|_{L^1} + C(1+t)\int_{\mathbf{R}^N}\langle v\rangle^2 M(v)dv \le C(1+t)\,.\end{aligned}$$

To estimate the second term in the right hand side of (7.23), we let

$$G(v,t)=\frac{1}{1-e^{-t-1}}g(v,t)=f(v,t)+\zeta(t)M(v)\,,\quad \zeta(t)=\frac{e^{-t-1}}{1-e^{-t-1}}\,. \tag{7.24}$$

Then ( recalling $M'M'_*=MM_*$)

$$f'f'_*-ff_*-(G'G'_*-GG_*)=\zeta(t)\Big(Mf_*+fM_*-M'f'_*-M'_*f'\Big)\,. \tag{7.25}$$

and

$$G'G'_*-GG_*=(1-e^{-t-1})^{-2}(g'g'_*-gg_*)\,,\quad (1-e^{-t-1})\zeta(t)=e^{-t-1} \tag{7.26}$$

and so we compute

$$\begin{aligned}&(1-e^{-t-1})\int_{\mathbf{R}^N}Q(f)(v,t)\log g(v,t))\,dv\\ =\;& e^{-t-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}}B(Mf_*+fM_*-M'f'_*-M'_*f')\log g\,d\sigma dv_* dv\\ -\;& (1-e^{-t-1})^{-1}D(g(t))\,.\end{aligned} \tag{7.27}$$

Next using (7.17), (7.16) (for $\log^+(1/g)$ and $\log^+ g$) we compute

$$
\begin{aligned}
&\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(Mf_* + fM_* - M'f'_* - M'_*f')\log g\, d\sigma dv_* dv\\
&= \int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} BfM_*\left(\log g + \log g_* + \log(1/g') + \log(1/g'_*)\right) d\sigma dv_* dv\\
&\le A_0\int_{\mathbf{R}^N\times\mathbf{R}^N}\frac{1}{|v-v_*|^{|\gamma|}} fM_*\left(\log^+ f + \log^+ f_* + C(1+t)(\langle v\rangle^2 + \langle v_*\rangle^2)\right) dv_* dv\\
&= A_0\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{M_*}{|v-v_*|^{|\gamma|}}dv_*\right) f\log^+ f dv + A_0\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{1}{|v-v_*|^{|\gamma|}}M_*\log^+ f_* dv_*\right) f dv\\
&+C(1+t)\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{M_*}{|v-v_*|^{|\gamma|}}dv_*\right)\langle v\rangle^2 f dv + C(1+t)\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{\langle v_*\rangle^2 M_*}{|v-v_*|^{|\gamma|}}dv_*\right) f\, dv\\
&\le C(1+t) + C\|f(t)\log^+ f(t)\|_{L^1} \le C(1+t)\,.
\end{aligned}
$$

Here we used the fact that $\langle\cdot\rangle^s M \in L^\infty\cap L^1(\mathbf{R}^N)$ and $\log^+ f \in L^p\cap L^1(\mathbf{R}^N)$ for all $1<p<\infty$.

Summarizing the above estimates we obtain (7.19).

Proof of (7.20): This is the main estimate in this step. To do this we shall use the functions $\Phi_R(v), S_R(v, v_*, v', v'_*)$ defined in (7.1)-(7.2) for $\Phi_R(v) = \min\{\langle v\rangle^k,\ R\}$ with $k = s-2$.

Let

$$
\begin{aligned}
H_{k,R}(g(t)) &= \int_{\mathbf{R}^N}\Phi_R(v)g(v,t)\log g(v,t)dv\,,\\
H^+_{k,R}(g(t)) &= \int_{\mathbf{R}^N}\Phi_R(v)g(v,t)\log^+ g(v,t)dv\\
H^-_{k,R}(g(t)) &= \int_{\mathbf{R}^N}\Phi_R(v)g(v,t)\log^+(1/g(v,t))dv\,.
\end{aligned}
$$

To prove (7.20), it suffices to prove that

$$
H_{k,R}(g(t)) \le C(1+t)^2 + C\int_0^t e^{-\tau}H^+_{k,R}(g(\tau))d\tau \qquad \forall\, t\ge 0 \tag{7.28}
$$

where and below all the constants $C$ do not depend on $R$ and $\|B\|_{L^\infty}$. In fact if (7.28) holds true, then using $H^+_{k,R}(g(t)) = H_{k,R}(g(t)) + H^-_{k,R}(g(t))$ and the obvious estimate $H^-_{k,R}(g(t)) \le C(1+t)\max\{\|f(t)\|_{L^1_{k+2}},\ \|M\|_{L^1_{k+2}}\} \le C(1+t)^2$ we get

$$
H^+_{k,R}(g(t)) \le C(1+t)^2 + C\int_0^t e^{-\tau}H^+_{k,R}(g(\tau))d\tau\,, \quad \forall\, t\ge 0\,.
$$

So applying Gronwall's Lemma gives

$$
H^+_{k,R}(g(t)) \le C(1+t)^2\exp\left(\int_0^t Ce^{-\tau}d\tau\right) \le C(1+t)^2\,.
$$

Then letting $R\to\infty$ leads to (7.20) by Fatou's Lemma.

As usual, for notational convenience we also denote without confusion that

$$
S_R = S_R(v, v_*, v', v'_*)\,, \quad \Phi_R = \Phi_R(v)\,, \quad \Phi'_R = \Phi_R(v')\,, \quad (\Phi_R)'_* = \Phi_R(v'_*)\,, \quad \text{etc.}
$$

Now we begin to prove (7.28). It has been shown in the above that there are no problem of integrability and the function $t \mapsto H_{k,R}(g(t))$ is absolutely continuous on any finite intervals. We compute using (7.22) with $\varphi = \Phi_R$ that

$$
\begin{aligned}
\frac{d}{dt}H_{k,R}(g(t)) = e^{-t-1} & \int_{\mathbf{R}^N} \Phi_R(v)(f(v,t) - M(v))dv \\
&+(1-e^{-t-1})\int_{\mathbf{R}^N} \Phi_R(v)Q(f)(v,t)dv \\
&+e^{-t-1}\int_{\mathbf{R}^N} \Phi_R(v)(f(v,t) - M(v))\log g(v,t)dv \\
&+(1-e^{-t-1})\int_{\mathbf{R}^N} \Phi_R(v)Q(f)(v,t)\log g(v,t)dv \\
&:= I^{(1)}_{k,R}(t) + I^{(2)}_{k,R} + I^{(3)}_{k,R}(t) + I^{(4)}_{k,R}(t)\,.
\end{aligned}
$$

To estimate these terms we shall use the following inequality:

$$
|\Phi_R(w) - S_R(v,v_*,v',v'_*)| \le k(\langle v\rangle^2 + \langle v_*\rangle^2)^{(k-|\gamma|)/2}|v-v_*|^{|\gamma|} \quad \forall\, w \in \{v, v_*, v', v'_*\}\,. \tag{7.29}
$$

Now we are going to estimate $I^{(i)}_{k,R}\,(i = 1,2,3,4)$. Let us emphasize again that **all constants $C$ are independent of $R$ and $\|B\|_{L^\infty(\mathbf{R}^N\times\mathbf{S}^{N-1})}$.**

The first one is easy: By moment estimate we have

$$
I^{(1)}_{k,R}(t) \le e^{-t-1}\|f(t)\|_{L^1_k} \le C(1+t)e^{-t}\,.
$$

For the second term we use the vanishing property

$$
\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(v-v_*,\sigma)S_R(v,v_*,v',v'_*)(f'f'_* - ff_*)d\sigma dv_* dv = 0 \tag{7.30}
$$

which is due to the symmetry

$$
S_R(v,v_*,v',v'_*) = S_R(v_*,v,v'_*,v') = S_R(v',v'_*,v,v_*)\,, \quad \text{etc.}
$$

Then using (7.29) we compute

$$
\begin{aligned}
I^{(2)}_{k,R}(t) &= (1-e^{-t-1})\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(\Phi_R - S_R)(f'f'_* - ff_*)d\sigma dv_* dv \\
&\le C\|f(t)\|_{L^1_{k-|\gamma|}} \le C(1+t)\,.
\end{aligned}
$$

For the third term $I^{(3)}_{k,R}(t)$ we use the control

$$
f(v,t) \le \frac{1}{1-e^{-t-1}}g(v,t)
$$

to see that

$$
\begin{aligned}
I^{(3)}_{k,R}(t) &\le e^{-t-1}\int_{\mathbf{R}^N} \Phi_R f\log^+ g dv + e^{-t-1}\int_{\mathbf{R}^N} \Phi_R M\log^+(1/g)dvd\tau \\
&\le e^{-t}H^+_{k,R}(g(t)) + C(1+t)e^{-t}\,.
\end{aligned}
$$

The estimate of the last term $I^{(4)}_{k,R}(t)$ is the key part of this section. One will see that in this estimate the term $H^+_{k,R}(g(t))$ can not be avoided and this is why we introduce the decay weight $e^{-t-1}$ rather than the equal weight $1/2$.

Inserting the function $G'G'_* - GG_* = (1-e^{-t-1})^{-2}(g'g'_* - gg_*)$ (see (7.24),(7.26)) we have

$$
\begin{aligned}
I^{(4)}_{k,R}(t) &= (1-e^{-t-1})\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\Phi_R(f'f'_* - ff_*)\log g dv \\
&= (1-e^{-t-1})\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\Phi_R\Big(f'f'_* - ff_* - (G'G'_* - GG_*)\Big)\log g dv \\
&\quad +(1-e^{-t-1})^{-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(\Phi_R - S_R)(g'g'_* - gg_*)\log g\, dv \\
&\quad +(1-e^{-\tau-1})^{-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} BS_R(g'g'_* - gg_*)\log g\, dv \\
&:= I^{(4,1)}_{k,R}(t) + I^{(4,2)}_{k,R}(t) + I^{(4,3)}_{k,R}(t)\,.
\end{aligned}
$$

Further estimate using (7.25) and $(1-e^{-t-1})\zeta(t) = e^{-t-1}$:

$$
\begin{aligned}
I^{(4,1)}_{k,R}(t) &\le e^{-t-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\Phi_R M f_*\log^+ g\, d\sigma dv_* dv \\
&\quad + e^{-t-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\Phi_R f M_*\log^+ g\, d\sigma dv_* dv \\
&\quad + e^{-t-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\Phi_R(M'f'_* + f'M'_*)\log^+(1/g) d\sigma dv_* dv \\
&:= I^{(4,1,1)}_{k,R}(t) + I^{(4,1,2)}_{k,R}(t) + I^{(4,1,3)}_{k,R}(t)\,,
\end{aligned}
$$

$$
\begin{aligned}
I^{(4,1,1)}_{k,R}(t) &\le A_0 e^{-t-1}\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{\langle v\rangle^k M}{|v-v_*|^{|\gamma|}}\log^+ g\, dv\right) f_* dv_* \\
&\le Ce^{-t-1}\|f(t)\|_{L^1} \le C\,,
\end{aligned}
$$

and (using $f(v,t) \le \frac{1}{1-e^{-t-1}} g(v,t)$)

$$
\begin{aligned}
I^{(4,1,2)}_{k,R}(t) &\le A_0\frac{e^{-t-1}}{1-e^{-t-1}}\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{M_* dv_*}{|v-v_*|^{|\gamma|}}\right)\Phi_R\, g\log^+ g\, dv \\
&\le Ce^{-t}\int_{\mathbf{R}^N}\Phi_R\, g\log^+ g\, dv = Ce^{-t}H^+_{k,R}(g(t))\,.
\end{aligned}
$$

For $I^{(4,1,3)}_{k,R}(t)$ we change variables and use inequality

$$
\Phi'_R\log^+(1/g') + (\Phi_R)'_*\log^+(1/g'_*) \le C(1+t)\langle v\rangle^{k+2}\langle v_*\rangle^{k+2}
$$

recalling $k+2 = s$ to get

$$
\begin{aligned}
I^{(4,1,3)}_{k,R}(t) &= e^{-t-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B\Big(\Phi'_R\log^+(1/g') + (\Phi_R)'_*\log^+(1/g'_*)\Big)M_* f\, d\sigma dv_* dv \\
&\le CA_0(1+t)e^{-t-1}\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{\langle v_*\rangle^s M_*}{|v-v_*|^{|\gamma|}}dv_*\right)\langle v\rangle^s f dv \\
&\le C(1+t)e^{-t-1}\|f(\tau)\|_{L^1_s} \le C(1+t)^2 e^{-t}\,.
\end{aligned}
$$

Thus
$$I_{k,R}^{(4,1)}(t) \le Ce^{-t}H_{k,R}^{+}(g(t)) + C\,.$$

**Estimate of $I_{k,R}^{(4,2)}(t)$:** Neglecting negative parts we have

$$\begin{aligned}
&I_{k,R}^{(4,2)}(t) = (1-e^{-t-1})^{-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(\Phi_R' - S_R)gg_*\log g'\,d\sigma dv_* dv\\
&+(1-e^{-t-1})^{-1}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(S_R-\Phi_R)gg_*\log g\,d\sigma dv_* dv\\
&\le 2\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(\Phi_R'-S_R)^+ gg_*\log^+ g'\,d\sigma dv_* dv\\
&+2\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(S_R-\Phi_R')^+ gg_*\log^+(1/g')\,d\sigma dv_* dv\\
&+2\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(S_R-\Phi_R)^+ gg_*\log^+ g\,d\sigma dv_* dv\\
&+2\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} B(\Phi_R-S_R)^+ gg_*\log^+(1/g)\,d\sigma dv_* dv\\
&:= I_{k,R}^{(4,2,1)}(t)+I_{k,R}^{(4,2,2)}(t)+I_{k,R}^{(4,2,3)}(t)+I_{k,R}^{(4,2,4)}(t)\,.
\end{aligned}$$

Using $B(v-v_*,\sigma)\le b(\cos\theta)|v-v_*|^{-|\gamma|}$ and Lemma 7.1 with $\beta=|\gamma|$ we have
$$I_{k,R}^{(4,2,1)}(t)+I_{k,R}^{(4,2,3)}(t)\le C\left(\|g\log^+ g\|_{L^1}+\|g\|_{L^1}\right)\|g\|_{L^1_{k-|\gamma|}}\le C(1+t)\,,$$

while using (7.29) and $\log^+(1/g')\le C(1+t)(\langle v\rangle^2+\langle v_*\rangle^2)$ gives
$$I_{k,R}^{(4,2,2)}(t)+I_{k,R}^{(4,2,4)}(t)\le C(1+t)\|g(t)\|_{L^1_{s-|\gamma|}}\,.$$
Since $\|g(t)\|_{L^1_{s-|\gamma|}}\le\|f(t)\|_{L^1_{s-|\gamma|}}+C$, it follows that
$$I_{k,R}^{(4,2)}(t)\le C(1+t)(\|f(t)\|_{L^1_{s-|\gamma|}}+1)\,.$$

The last term $I_{k,R}^{(4,3)}(t)$ is negative which is due to the total symmetry of $S_R(v,v_*,v',v_*')$: Using classical derivation one has
$$I_{k,R}^{(4,3)}(t)=-(1-e^{-t-1})^{-1}D_{S_R}(g(t))\le 0$$
where
$$D_{S_R}(g(t))=\frac{1}{4}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} BS_R(g'g_*'-gg_*)\log\left(\frac{g'g_*'}{gg_*}\right)\,d\sigma dv_* dv\ge 0\,.$$
Summarizing the above we obtain
$$I_{k,R}^{(4)}(t)\le C(1+t)(\|f(t)\|_{L^1_{s-|\gamma|}}+1)+Ce^{-t}H_{k,R}^{+}(g(t))\,.$$

And therefore collecting all estimates of $I_{k,R}^{(1)}(t)$, $I_{k,R}^{(2)}(t)$, $I_{k,R}^{(3)}(t)$, $I_{k,R}^{(4)}(t)$ we obtain
$$\frac{d}{dt}H_{k,R}(g(t))\le C(1+t)(\|f(t)\|_{L^1_{s-|\gamma|}}+1)+Ce^{-t}H_{k,R}^{+}(g(t))\quad\text{a.e.}\quad t\ge 0\,.$$

Since $\int_0^t \|f(\tau)\|_{L^1_{s-|\gamma|}} d\tau \le C(1+t)$ and $H_{k,R}(g(0)) \le \|f_0 \log^+ f_0\|_{L^1_2}$, it follows that

$$\begin{aligned} H_{k,R}(g(t)) &\le C + C\int_0^t (1+\tau)(\|f(\tau)\|_{L^1_{s-|\gamma|}} + 1)d\tau + C\int_0^t e^{-\tau} H^+_{k,R}(g(\tau))d\tau \\ &\le C(1+t)^2 + C\int_0^t e^{-\tau} H^+_{k,R}(g(\tau))d\tau\,. \end{aligned}$$

This proves (7.28) and finishes the Step 1.

**Step 2**. The method of proving (7.14) is to establish an inequality between $H(g(t))$ and $D(g(t))$ as has been done in the case of hard potentials, see e.g. [4, 5, 19]. [ Note that $g(v,t)$ is not a solution of Eq.(B), but it is better than a solution in the present sense ...] We shall prove that

$$D(g(t)) \ge c(1+t)^{-2\varepsilon}[H(g(t)|M)]^{1+\varepsilon} \qquad \forall\, t \ge 0 \tag{7.31}$$

where

$$\varepsilon = \frac{2+|\gamma|}{k-2} < \frac{1}{2}\,, \quad k = s-2 \;\; (> 6+2|\gamma|\,)\,.$$

If (7.31) holds true, then by the differential inequality (7.19) we get

$$\frac{d}{dt}H(g(t)|M) \le -c(1+t)^{-2\varepsilon}[H(g(t)|M)]^{1+\varepsilon} + C(1+t)e^{-t} \qquad \text{a.e.} \quad t \in [0,\infty)\,.$$

Applying Lemma 7.2 we then obtain (with $\alpha = \frac{1-2\varepsilon}{\varepsilon} \ge 4\lambda$)

$$H(g(t)|M) \le C(1+t)^{-\alpha} \le C(1+t)^{-4\lambda}$$

and hence (7.14) follows from (7.18).

To prove (7.31), we consider

$$\mathcal{D}_k(g) = \frac{1}{4}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} (1+|v-v_*|^2)^{k/2}(g'g'_* - gg_*)\log\left(\frac{g'g'_*}{gg_*}\right) d\sigma dv dv_*$$

and make use of Villani's inequality (see Lemma 7.3 and note that $H(g(t)|M) \le H(f(t)|M) \le H(f_0|M)$):

$$C_{H_0} H(g(t)|M) \le \mathcal{D}_2(g(t)) \tag{7.32}$$

where

$$C_{H_0} = \frac{|\mathbf{S}^{N-1}|}{4(2N+1)} \inf_{f\in\mathcal{H}_0}(N - T^*_f) > 0\,, \quad H_0 = H(f_0|M)\,.$$

By assumption $K_*(1+|z|^2)^{-|\gamma|/2} \le B(z,\sigma)$ and writing $\frac{k}{2} = (1+\frac{|\gamma|}{2(1+\varepsilon)})\cdot\frac{1+\varepsilon}{\varepsilon}$ we have

$$K_*^{\frac{1}{1+\varepsilon}}(1+|z|^2) \le [B(z,\sigma)]^{\frac{1}{1+\varepsilon}}\left((1+|z|^2)^{k/2}\right)^{\frac{\varepsilon}{1+\varepsilon}}.$$

So by Hölder's inequality

$$K_*[\mathcal{D}_2(g(t))]^{1+\varepsilon} \le D(g(t))[\mathcal{D}_k(g(t))]^{\varepsilon}\,, \quad t \ge 0\,. \tag{7.33}$$

Now we prove that

$$\mathcal{D}_k(g(t)) \le C(1+t)^2\,. \tag{7.34}$$

By symmetry we have

$$\mathcal{D}_k(g) = \frac{1}{2}\int_{\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{S}^{N-1}} (1+|v-v_*|^2)^{k/2} 1_{\{gg_*\ge g'g'_*\}}(gg_*-g'g'_*)\log\left(\frac{gg_*}{g'g'_*}\right) d\sigma dv dv_*\,.$$

Note that using (7.17) gives

$$\log(\frac{1}{g'g'_*}) \le C(1+t)(\langle v\rangle^2+\langle v_*\rangle^2)\,.$$

So if $gg_*\ge g'g'_*$ then

$$\begin{aligned}(gg_*-g'g'_*)\log\left(\frac{gg_*}{g'g'_*}\right) &\le gg_*\left(\log(gg_*)+\log\left(\frac{1}{g'g'_*}\right)\right)\\ &\le gg_*\log^+ g + gg_*\log^+ g_* + Cgg_*(1+t)(\langle v\rangle^2+\langle v_*\rangle^2)\,.\end{aligned}$$

Since $(1+|v-v_*|^2)^{k/2} \le 2^{k-1}(\langle v\rangle^k+\langle v_*\rangle^k)$, it follows from the main estimate (7.20), $\|g(t)\log^+ g(t)\|_{L^1}\le \|f(t)\log^+ f(t)\|_{L^1}\le C$, $\|g(t)\|_{L^1_s}\le C(1+t)$, and $k+2=s$ that

$$\begin{aligned}&\mathcal{D}_k(g(t)) \le C\int_{\mathbf{R}^N\times\mathbf{R}^N}(\langle v\rangle^k+\langle v_*\rangle^k) g\log^+(g) g_* dv dv_*\\ &+C(1+t)\int_{\mathbf{R}^N\times\mathbf{R}^N}(\langle v\rangle^{k+2}+\langle v_*\rangle^{k+2}) gg_* dv dv_*\\ &\le C\Big(\|g(t)\log^+ g(t)\|_{L^1_k}+\|g(t)\log^+ g(t)\|_{L^1}\|g(t)\|_{L^1_k}+(1+t)\|g(t)\|_{L^1_s}\Big)\\ &\le C(1+t)^2\,.\end{aligned}$$

This proves (7.34). Combining (7.32)-(7.34) we obtain

$$K_*[C_{H_0}H(g(t)|M)]^{1+\varepsilon}\le C(1+t)^{2\varepsilon}D(g(t))$$

which gives (7.31).

**Step 3**. Let $B(z,\sigma)$ be given in the theorem. We shall use approximate solutions. Let

$$B_n(z,\sigma)=\min\{B(z,\sigma)\,,\,n\}\,,\quad n\ge K_*\,.$$

It is obvious that for all $n\ge K_*$, $B_n(z,\sigma)\ge K_*(1+|z|^2)^{-|\gamma|/2}$. For each $n\ge K_*$, let $f^n\in C^1([0,\infty);L^1(\mathbf{R}^N))\cap L^\infty([0,\infty);L^1_2\cap L^1\log L(\mathbf{R}^N))$ be the unique mild solution of Eq.(B) with the kernel $B_n$ and $f^n|_{t=0}=f_0$ and $f^n$ has all properties as listed in Step 1 and Step 2. In particular $f^n$ satisfies for all $n\ge K_*$ and all $t\ge 0$

$$H(f^n(t)|M)\le C(1+t)^{-4\lambda}$$

with the same constants $\lambda>0$ and $0<C<\infty$ (which is of course independent of $n$). As argued in the proof of existence of weak solutions, there exists a subsequence $\{f^{n_k}\}_{k=1}^\infty$ of $\{f^n\}$ and a weak solution $f$ of Eq.(B) satisfying $f|_{t=0}=f_0$, such that

$$\forall\, t\ge 0\,,\quad f^{n_k}(\cdot,t)\rightharpoonup f(\cdot,t)\ \ (k\to\infty)\quad \text{weakly in}\quad L^1(\mathbf{R}^N)\,.$$

Then by convexity and weak convergence, we obtain

$$H(f(t)|M)\le \liminf_{k\to\infty} H(f^{n_k}(t)|M)\le C(1+t)^{-4\lambda}\quad \forall\, t\ge 0\,.$$

This completes the proof. $\Box$

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